Secondary Mathematics 3 Table of Contents

Polynomials

Series

Rational and Radical Expressions and Equations

Graphing Functions

Solving Equations and Inequalities

Building Functions

Logarithms

Unit 2 Cluster 4 (A.APR.1): Polynomials

Cluster 4: Perform arithmetic operations on polynomials
- 2.4 Polynomials are closed under addition, subtraction, and multiplication
- 2.4 Add, subtract, and multiply polynomials (**NO DIVISION**)

VOCABULARY

A term that does not have a variable is called a **constant.** For example the number 5 is a **constant** because it does not have a variable attached and will always have the value of 5.

A constant or a variable, or a product of a constant and a variable is called a **term**. For example $2,\ x$, or $-3x^2$ are all terms.

Terms with the same variable to the same power are **like terms**. $2x^2$ and $-7x^2$ are like terms.

An expression formed by adding a finite number of unlike terms is called a **polynomial**. The variables can only be raised to positive integer exponents. $4x^3-6x^2+1$ is a polynomial, while $x^{\frac{3}{2}}-2x^{-1}+5$ is not a polynomial. **NOTE:** There are no square roots of variables, no fractional powers, and no variables in the denominator of any fractions.

A polynomial with only one term is called a **monomial** $\left(6x^4\right)$. A polynomial with two terms is called a **binomial** $\left(2x+1\right)$. A polynomial with three terms is called a **trinomial** $\left(5x^2-x+3\right)$**.**

Polynomials are in **standard (general) form** when written with exponents in descending order and the constant term last. For example $2x^4-5x^3+7x^2-x+3$ is in standard form.

The exponent of a term gives you the **degree** of the term. The term $-3x^2$ has degree two. For a polynomial, the value of the *largest exponent* is the **degree** of the whole polynomial. The polynomial $2x^4-5x^3+7x^2-x+3$ has degree 4.

When the term contains a variable and a number, the number part of a term is called the **coefficient**. $6x$ has a coefficient of 6 and $-x^2$ has a coefficient of -1.

The **leading coefficient** is the coefficient of the first term when the polynomial is written in standard form. 2 is the leading coefficient of $2x^4-5x^3+7x^2-x+3$.

General Polynomial: $$f(x)=\underbrace{\underbrace{a_n}_{\text{Leading Coefficient } a_n}x^{\overset{\text{Degree } n}{n}}}_{\text{Leading Term}}+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+\underbrace{a_0}_{\text{Constant}}$$

CLASSIFICATIONS OF POLYNOMIALS

Name	Form	Degree	Example
Zero	$f(x)=0$	None	$f(x)=0$
Constant	$f(x)=a, a\neq 0$	0	$f(x)=5$
Linear	$f(x)=ax+b$	1	$f(x)=-2x+1$
Quadratic	$f(x)=ax^2+bx+c$	2	$f(x)=3x^2+\frac{1}{2}x+\frac{7}{9}$
Cubic	$f(x)=ax^3+bx^2+cx+d$	3	$f(x)=x^3-3x^2$

Polynomial Operations

Addition/Subtraction: Combine like terms.

Example 1:

Horizontal Method	Vertical Method
$\left(2x^3-3x^2+4x-1\right)+\left(x^3+2x^2-5x+3\right)$ $\left(2x^3+x^3\right)+\left(-3x^2+2x^2\right)+\left(4x-5x\right)+\left(-1+3\right)$ $3x^3-x^2-x+2$	$\begin{array}{r} 2x^3-3x^2+4x-1 \\ +\quad x^3+2x^2-5x+3 \\ \hline 3x^3-\quad x^2-\ x+2 \end{array}$

Example 2:

Horizontal Method	Vertical Method
$\left(4x^2+3x-4\right)-\left(2x^3+x^2-x+2\right)$ $4x^2+3x-4-2x^3-x^2+x-2$ $-2x^3+3x^2+4x-6$	$\begin{array}{r} 4x^2+3x-4 \\ -\left(2x^3+x^2\ -\ x+2\right) \\ \hline -2x^3+3x^2+4x-6 \end{array}$

Practice Exercises A

Perform the required operations. Write your answers in standard form and determine if the result is a polynomial.

1. $\left(3x^2-4x+1\right)+\left(-x^2+x-9\right)$
2. $\left(5x^2+x^3+6\right)+\left(x^2+5-6x\right)$
3. $\left(x^2+1\right)+\left(-4x^2+5\right)$
4. $\left(5x^2-4x+1\right)-\left(8-x^2\right)$
5. $\left(-3+4n^2\right)-\left(5-2n^3\right)$
6. $\left(3t^2-8t+2\right)-\left(-3t^2+5t-7\right)$

7. $\left(7+2x-4x^2\right)+\left(-3x+x^2-5\right)$

8. $\left(-8x^2-3x+7\right)+\left(-x^3+6x^2-5\right)$

9. $\left(9x^3-5x^2+x\right)+\left(6x^2+5x-10\right)$

10. $12-\left(-5x^2+x-7\right)$

11. $\left(x-4x^2+7\right)-\left(-5x^2+5x-3\right)$

12. $\left(3x^2+4\right)-\left(x^2-5x+2\right)$

Multiplication: Multiply two binomials $(5x-7)(2x+9)$

<table>
<tr><th>Distributive (FOIL) Method</th><th>Box Method</th><th>Vertical Method</th></tr>
<tr>
<td>$(5x-7)(2x+9)$
$5x(2x+9)-7(2x+9)$
$10x^2+45x-14x-63$
combine like terms
$10x^2+31x-63$</td>
<td>
<table>
<tr><td></td><td>$5x$</td><td>-7</td></tr>
<tr><td>$2x$</td><td>$10x^2$</td><td>$-14x$</td></tr>
<tr><td>9</td><td>$45x$</td><td>-63</td></tr>
</table>
combine terms on the diagonals of the unshaded boxes(top right to lower left)
$10x^2+31x-63$</td>
<td>$$\begin{array}{rrr} & & 5x-7 \\ \times & & 2x+9 \\ \hline & & 45x-63 \\ 10x^2-14x & & \\ \hline 10x^2+31x-63 & & \end{array}$$</td>
</tr>
</table>

Multiplication: Multiply a binomial and a trinomial $(2x+3)\left(6x^2-7x-5\right)$

<table>
<tr><th>Distributive Method</th><th>Box Method</th><th>Vertical Method</th></tr>
<tr>
<td>$(2x+3)\left(6x^2-7x-5\right)$
$2x\left(6x^2-7x-5\right)+3\left(6x^2-7x-5\right)$
$\left(12x^3-14x^2-10x\right)+\left(18x^2-21x-15\right)$
$12x^3-14x^2-10x+18x^2-21x-15$
combine like terms
$12x^3+4x^2-31x-15$</td>
<td>
<table>
<tr><td></td><td>$6x^2$</td><td>$-7x$</td><td>-5</td></tr>
<tr><td>$2x$</td><td>$12x^3$</td><td>$-14x^2$</td><td>$-10x$</td></tr>
<tr><td>3</td><td>$18x^2$</td><td>$-21x$</td><td>-15</td></tr>
</table>
combine terms on the diagonals of the unshaded boxes(top right to lower left)
$12x^3+4x^2-31x-15$</td>
<td>$$\begin{array}{rrrr} & 6x^2 & -7x & -\ 5 \\ \times & & 2x & +\ 3 \\ \hline & 18x^2 & -21x & -15 \\ 12x^3 & -14x^2 & -10x & \\ \hline 12x^3 & +\ 4x^2 & -31x & -15 \end{array}$$</td>
</tr>
</table>

Multiplication: Multiply a trinomial and a trinomial $\left(2x^2+3x-1\right)\left(6x^2-7x-5\right)$

Distributive Method

$$\left(2x^2+3x-1\right)\left(6x^2-7x-5\right)$$

$$2x^2\left(6x^2-7x-5\right)+3x\left(6x^2-7x-5\right)-1\left(6x^2-7x-5\right)$$
$$\left(12x^4-14x^3-10x^2\right)+\left(18x^3-21x^2-15x\right)+\left(-6x^2+7x+5\right)$$
$$12x^4+4x^3-37x^2-8x+5$$

Box Method

	$6x^2$	$-7x$	-5
$2x^2$	$12x^4$	$-14x^3$	$-10x^2$
$3x$	$18x^3$	$-21x^2$	$-15x$
-1	$-6x^2$	$7x$	5

**combine terms on the diagonals of the unshaded boxes(top right to lower left)*

$$12x^4+4x^3-37x^2-8x+5$$

Vertical Method

$$\begin{array}{lr} & 2x^2+3x-1 \\ \times & 6x^2-7x-5 \\ \hline & 10x^2+15x+5 \\ & -14x^3-21x^2-15x \quad\quad \\ & 12x^4+18x^3-6x^2 \quad\quad\quad\quad\quad\quad \\ \hline & 12x^4+4x^3-37x^2-8x+5 \end{array}$$

Practice Exercises B

Perform the required operations. Write your answers in standard form and determine if the result is a polynomial.

1. $(6x-3)(-5x-6)$
2. $\left(3x-5\right)\left(x+2\right)$
3. $(7x+2)(10x+5)$
4. $\left(2x+3\right)\left(4x+1\right)$
5. $(-4x-5)(9x+8)$
6. $\left(3x-y\right)\left(3x+y\right)$
7. $\left(2x+7\right)^2$
8. $\left(3-5x\right)^2$
9. $\left(5x^3-1\right)^2$
10. $\left(2x^3-3y\right)\left(2x^3+3y\right)$
11. $\left(x^2-2x+3\right)\left(x+4\right)$
12. $\left(x^2+3x-2\right)\left(x-3\right)$

13. $(2x+7)(5x^2+4x+1)$

14. $\left(-3x^2-5\right)\left(x^2+7x+12\right)$

15. $(-9x-2)(-3x^2-8x-5)$

16. $(2x^2+4x+10)(3x-4)$

17. $\left(x^2+x-3\right)\left(x^2+x+1\right)$

18. $\left(2x^2-3x+1\right)\left(x^2+x+1\right)$

19. $(x^2-8x-1)(2x^2+10x+4)$

20. $\left(x^2-3x+7\right)\left(3x^2+5x-3\right)$

21. $\left(y^2+2y-3\right)\left(5y^2+3y+4\right)$

22. $(-3x^2+x+3)(2x^2+10x+4)$

23. $(4x^2+6x+1)(-5x^2-3x-6)$

24. $\left(4x^3-x^2+3x\right)\left(x^3+12x-3\right)$

YOU DECIDE

Are polynomials closed under addition, subtraction, multiplication? Justify your conclusion using the method of your choice.

Unit 2 Cluster 5: Polynomials
(A.APR.2, A.APR.3, F.IF.7c, and N.CN.9)

Cluster 5: Relationships between zeros and factors of polynomials
	2.5	Know and apply the Remainder Theorem
	2.5	Identify the zeros of polynomials and use the zeros to construct a rough graph.
Cluster 10: Analyze functions using different representations
	2.10	Graph polynomial functions identifying zeros and showing end behavior
Cluster 1: Use Complex Numbers in Polynomial Identities and Equations
	2.1	Know the Fundamental Theorem of Algebra

Remainder Theorem

For a polynomial $p(x)$ and a number *a*, the remainder when dividing by $x-a$ is $p(a)$, so $p(a)=0$ if and only if $x-a$ is a factor of $p(x)$.

Example 1:

Is $x+5$ a factor of $f(x)=3x^2+14x-5$?

$x-a$ $x+5=x-(-5)$ $a=-5$	Identify *a*.
$f(x)=3x^2+14x-5$ $f(-5)=3(-5)^2+14(-5)-5$ $=75-70-5$ $=0$	Substitute *a* in for *x*. Simplify
Since $f(-5)=0$ the binomial $x+5$ is a factor of $f(x)=3x^2+14x-5$.	
NOTE: If we factored $f(x)=3x^2+14x-5$, the result would be $f(x)=(3x-1)(x+5)$.	

Example 2:

Is $x-3$ a factor of $f(x)=2x^2-7x-4$?

$x-a$ $x-3$ $a=3$	Identify *a*.
$f(x)=2x^2-7x-4$ $f(3)=2(3)^2-7(3)-4$ $=18-21-4$ $=-7$	Substitute *a* in for *x*. Simplify
Since $f(3)\neq 0$ the binomial $x-3$ is not a factor of $f(x)=2x^2-7x-4$.	
NOTE: If we factored $f(x)=2x^2-7x-4$, the result would be $f(x)=(2x+1)(x-4)$.	

Example 3:

Is $x+2$ a factor of $f(x)=x^3-3x^2-6x+8$?

$x-a$ $x+2=x-(-2)$ $a=-2$	Identify *a*.
$f(x)=x^3-3x^2-6x+8$ $f(-2)=(-2)^3-3(-2)^2-6(-2)+8$ $=-8-12+12+8$ $=0$	Substitute *a* in for *x*. Simplify
Since $f(-2)=0$ the binomial $x+2$ is a factor of $f(x)=x^3-3x^2-6x+8$.	
NOTE: If we factored $f(x)=x^3-3x^2-6x+8$, the result would be $f(x)=(x+2)(x-1)(x-4)$.	

Practice Exercises A

For the given polynomials determine which of the binomials listed are factors.

1. $f(x)=-2x^2+15x-7$
   a. $x+1$
   b. $x-7$
   c. $x-2$

2. $f(x)=3x^2-7x-6$
   a. $x-3$
   b. $x+2$
   c. $x+1$

3. $f(x)=x^3+3x^2-4x-12$
   a. $x+2$
   b. $x-2$
   c. $x+3$

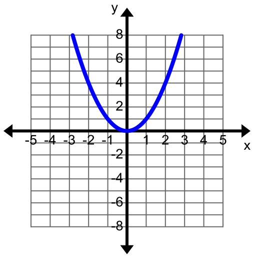
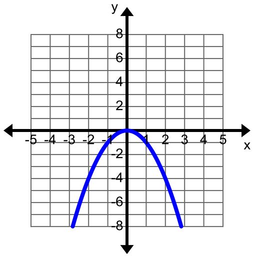
4. $f(x)=2x^3+15x^2+22x-15$
   a. $x+3$
   b. $x+5$
   c. $x-3$

Graphing Polynomials

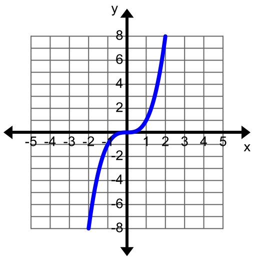
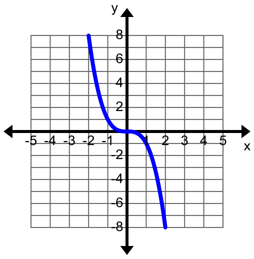
Using a graphing calculator it is easy to see the end behavior, identify the zeros, and the basic shape of any polynomial.

Power Function Graphs and End Behavior

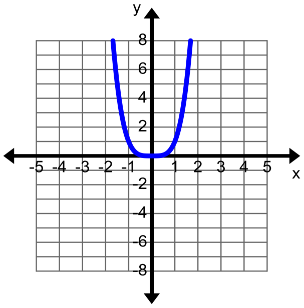
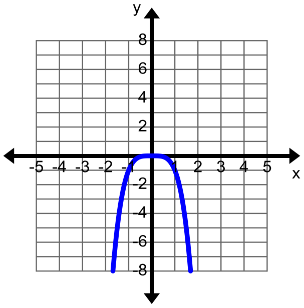
Quadratic (2^{nd} Degree)

Function	Graph	Left End Behavior	Right End Behavior
$f(x) = x^2$		$\lim_{x\to-\infty} f(x) = +\infty$	$\lim_{x\to\infty} f(x) = +\infty$
$f(x) = -x^2$		$\lim_{x\to-\infty} f(x) = -\infty$	$\lim_{x\to\infty} f(x) = -\infty$

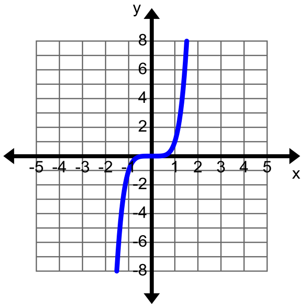
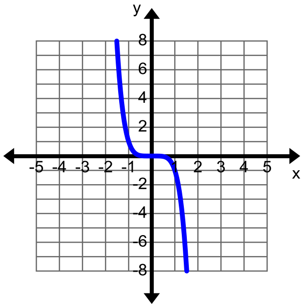
Cubic (3^{rd} Degree)

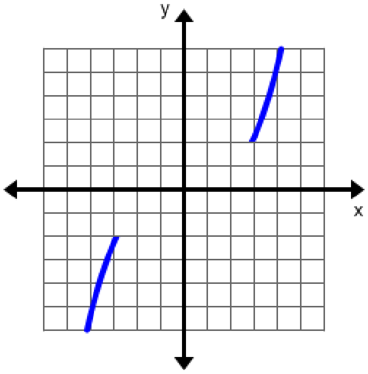
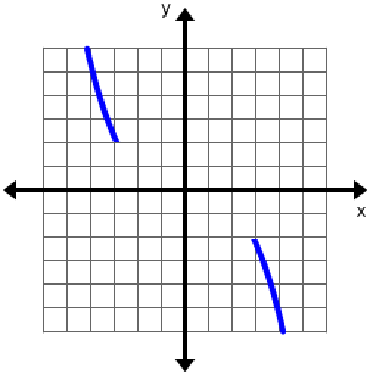
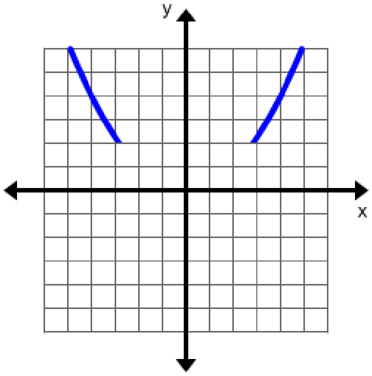
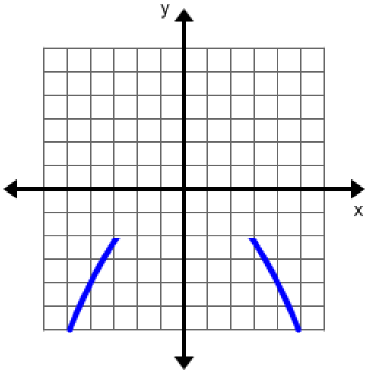
Function	Graph	Left End Behavior	Right End Behavior
$f(x) = x^3$		$\lim_{x\to-\infty} f(x) = -\infty$	$\lim_{x\to\infty} f(x) = +\infty$
$f(x) = -x^3$		$\lim_{x\to-\infty} f(x) = +\infty$	$\lim_{x\to\infty} f(x) = -\infty$

Quartic (4th Degree)

Function	Graph	Left End Behavior	Right End Behavior
$f(x) = x^4$		$\lim_{x\to-\infty} f(x) = +\infty$	$\lim_{x\to\infty} f(x) = +\infty$
$f(x) = -x^4$		$\lim_{x\to-\infty} f(x) = -\infty$	$\lim_{x\to\infty} f(x) = -\infty$

Quintic (5th Degree)

Function	Graph	Left End Behavior	Right End Behavior
$f(x) = x^5$		$\lim_{x\to-\infty} f(x) = -\infty$	$\lim_{x\to\infty} f(x) = +\infty$
$f(x) = -x^5$		$\lim_{x\to-\infty} f(x) = +\infty$	$\lim_{x\to\infty} f(x) = -\infty$

<table>
<tr><td colspan="2">Generalizing Polynomial End Behavior $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$</td></tr>
<tr><td colspan="2"><i>n</i> is odd</td></tr>
<tr><td>$a_n>0$

$\lim_{x\to-\infty} f(x)=-\infty$ $\lim_{x\to\infty} f(x)=+\infty$</td><td>$a_n<0$

$\lim_{x\to-\infty} f(x)=+\infty$ $\lim_{x\to\infty} f(x)=-\infty$</td></tr>
<tr><td colspan="2"><i>n</i> is even</td></tr>
<tr><td>$a_n>0$

$\lim_{x\to-\infty} f(x)=+\infty$ $\lim_{x\to\infty} f(x)=+\infty$</td><td>$a_n<0$

$\lim_{x\to-\infty} f(x)=-\infty$ $\lim_{x\to\infty} f(x)=-\infty$</td></tr>
</table>

Example 4: Without graphing, determine the end behavior of each polynomial.

a. $f(x)=-x^3+6x^2-5x+7$

b. $f(x)=2x^4+6x^2+7$

a. $f(x)=-x^3+6x^2-5x+7$ Left end behavior: $\lim_{x\to-\infty} f(x)=+\infty$ Right end behavior: $\lim_{x\to\infty} f(x)=-\infty$	$n=3$(odd) and $a_n=-1<0$

b. $f(x)=2x^4+6x^2+7$ Left end behavior: $\lim_{x\to-\infty} f(x)=+\infty$ Right end behavior: $\lim_{x\to\infty} f(x)=+\infty$	$n=4$(even) and $a_n=2>0$

Practice Exercises B

Without graphing, determine the end behavior of each polynomial.

1. $f\left(x\right)=2x^5+7x^3-4x$
2. $f\left(x\right)=-3x^6-8x^5+2x$
3. $f\left(x\right)=4x^7+5$
4. $f\left(x\right)=-10x^3-3x^2-5$
5. $f\left(x\right)=-6x^{10}+5x^4-5x^3+9$
6. $f\left(x\right)=8x^4+10x^3+3x-4$

Fundamental Theorem of Algebra

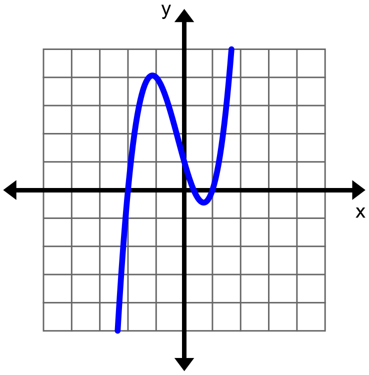
A polynomial function of degree $n>0$ has n complex zeros (every real number is a complex number i.e., $3=3+0i$). Some of the zeros may be repeated.

Example 5: Determine the number of zeros each of the following polynomials have.

a. $f(x)=3x^3+2x^2-7x+2$

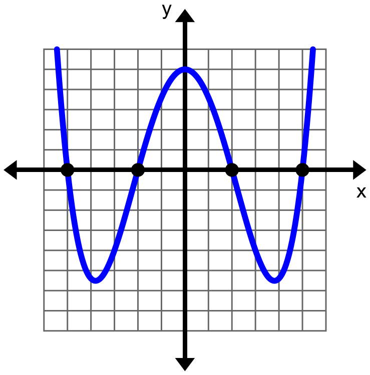
b. $g(x)=100+x^4-29x^2$

a. $f(x)$ is a 3rd degree polynomial function, therefore, the function will have three zeros.



You can see from the graph that the function crosses the *x*-axis three times.

b. $g(x)$ is a 4th degree polynomial function, therefore, the function will have four zeros.



You can see from the graph that the function crosses the *x*-axis four times.

Practice Exercises C

Without graphing, determine the number of zeros for each of the following polynomials.

1. $f(x) = 2x^4 - 5x^3 - 26x^2 - x + 30$
2. $f(x) = 3x^3 + x^2 - 62x + 40$
3. $f(x) = -4x^5 + 52x^3 - 144x$
4. $f(x) = 2x^4 + 5x^3 - 35x^2 - 80x + 48$
5. $f(x) = -x^6 + 14x^4 - 49x^2 + 36$
6. $f(x) = 5x^7 - 70x^5 + 245x^3 - 180x + 5$

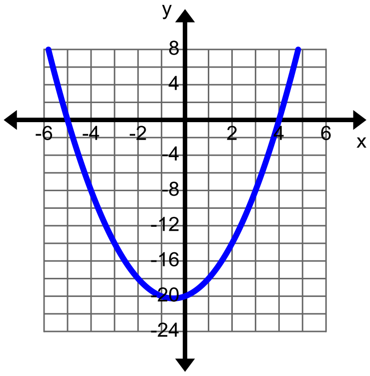
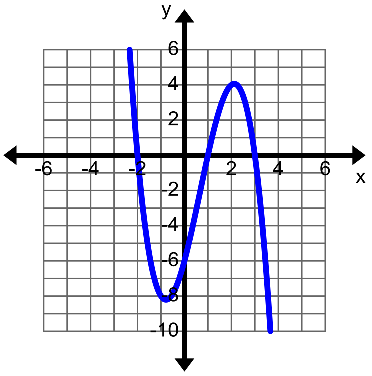
Example 6: Use technology to graph the polynomial. Identify its zeros and end behavior.

a. $f(x) = x^2 - x - 20$
$f(x) = (x-4)(x+5)$

b. $f(x) = -x^3 + 2x^2 + 5x - 6$
$f(x) = -(x-3)(x+2)(x-1)$

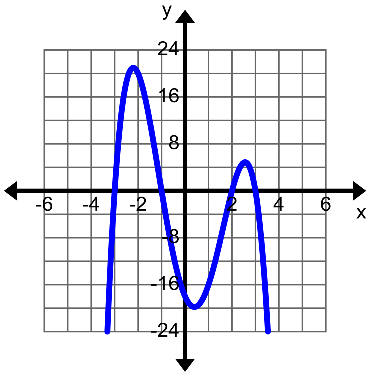
c. $f(x) = -x^4 + x^3 + 11x^2 - 9x - 18$
$f(x) = -(x^2-9)(x+1)(x-2)$

d. $f(x) = x^5 + 3x^4 - 20x^3 - 60x^2 + 64x + 192$
$f(x) = (x^2-4)(x+3)(x-4)(x+4)$

a. $f(x) = x^2 - x - 20$ $f(x) = (x-4)(x+5)$  The zeros are: $(-5,0)$ and $(4,0)$ Left end behavior: $\lim\limits_{x\to-\infty} f(x) = +\infty$ Right end behavior: $\lim\limits_{x\to\infty} f(x) = +\infty$	b. $f(x) = -x^3 + 2x^2 + 5x - 6$ $f(x) = -(x-3)(x+2)(x-1)$  The zeros are: $(-2,0)$, $(1,0)$, and $(3,0)$ Left end behavior: $\lim\limits_{x\to-\infty} f(x) = +\infty$ Right end behavior: $\lim\limits_{x\to\infty} f(x) = -\infty$

c. $f(x) = -x^4 + x^3 + 11x^2 - 9x - 18$

$f(x) = -(x^2 - 9)(x+1)(x-2)$



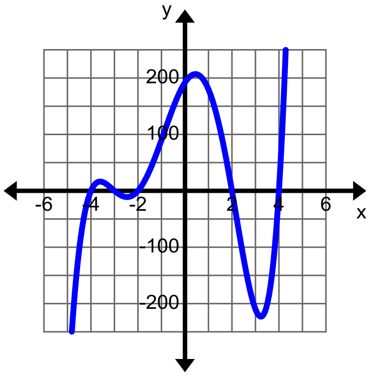
The zeros are: $(-3,0)$, $(-1,0)$, $(2,0)$, $(3,0)$

Left end behavior: $\lim\limits_{x\to-\infty} f(x) = -\infty$

Right end behavior: $\lim\limits_{x\to\infty} f(x) = -\infty$

d. $f(x) = x^5 + 3x^4 - 20x^3 - 60x^2 + 64x + 192$

$f(x) = (x^2 - 4)(x+3)(x-4)(x+4)$



The zeros are: $(-4,0)$, $(-3,0)$, $(-2,0)$, $(2,0)$, $(4,0)$

Left end behavior: $\lim\limits_{x\to-\infty} f(x) = -\infty$

Right end behavior: $\lim\limits_{x\to\infty} f(x) = +\infty$

Practice Exercises D

Use technology to graph the polynomial. Identify its zeros and end behavior.

1. $f(x) = (x+9)(x-10)(x-6)(x+2)$
   $f(x) = x^4 - 5x^3 - 98x^2 + 372x + 1080$

2. $f(x) = (x^2 - 9)(x+4)$
   $f(x) = x^3 + 4x^2 - 9x - 36$

3. $f(x) = (-3x)(x^2 - 1)(x^2 - 25)$
   $f(x) = -3x^5 + 78x^3 - 75x$

4. $f(x) = (x^2 - 1)(x^2 - 4)(x^2 - 25)$
   $f(x) = -x^6 + 30x^4 - 129x^2 + 100$

5. $f(x) = -2(x+3)(x-7)(x+6)$
   $f(x) = -2x^3 - 4x^2 + 90x + 252$

6. $f(x) = (5x)(x^2 - 1)(x^2 - 9)$
   $f(x) = 5x^5 - 50x^3 + 45x$

7. $f(x) = (2x+3)(x-1)(x-5)$
   $f(x) = 2x^3 - 8x^2 - 8x + 15$

8. $f(x) = (3x+2)(x-6)(x^2 - 4)$
   $f(x) = 3x^4 - 16x^3 - 24x^2 + 64x + 48$

Multiplicity of Zeros

Exploration:

Given: $f(x)=(x-3)(x+2)^2$

a. Identify the zeros of $f(x)$.

b. According to the Fundamental Theorem of Algebra how many zeros should $f(x)$ have?

c. Is there a difference between the number of zeros found in part a and the expected number of zeros found in part b?

a. $(x-3)(x+2)^2=0$	Set the function equal to zero.
$(x-3)(x+2)(x+2)=0$	Remember that $(x+2)^2=(x+2)(x+2)$.
$x-3=0$ $x+2=0$ $x+2=0$ $x=3$ $x=-2$ $x=-2$ $(3,0)$ $(-2,0)$ $(-2,0)$	Set each factor equal to zero. Solve for *x*. Write the zeros as ordered pairs.
The zeros of $f(x)$ are $(3,0)$ and $(-2,0)$ because $(-2,0)$ repeats.	

b. The expanded form of $f(x)$ is $f(x)=x^3+x^2-8x-12$. $f(x)$ is a 3rd degree polynomial function, therefore it should have three zeros.

c. No, three zeros were found, but only two were listed because one of the zeros repeats.

Multiplicity of Zeros: Recall that the Fundamental Theorem of Algebra states that zeros can be repeated. When a zero is repeated, the same factor occurs multiple times. We say the factor has a multiplicity of the number of times it is repeated. For instance, $(x-5)^4$ means that the zero $(5,0)$ is repeated four times and has a multiplicity of 4.

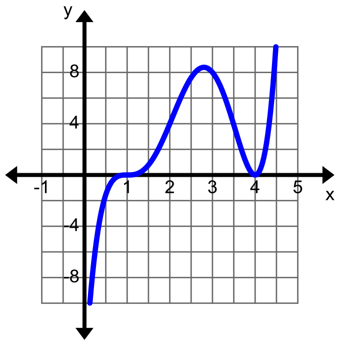
Example 7:

Use technology to graph the polynomial. Identify the zeros, their multiplicity, and determine whether they touch or cross the *x*-axis at each zero.

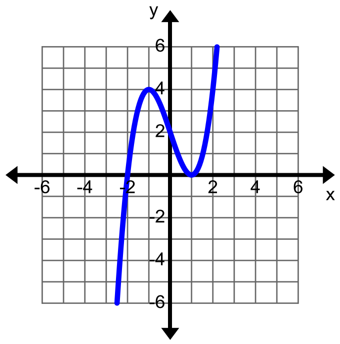
a. $f(x)=(x-4)^2(x-1)^3$

b. $f(x)=(x-1)^2(x+2)$

a.



The zeros are $(1,0)$ and $(4,0)$. The zero $(1,0)$ has multiplicity three and it crosses the *x*-axis. The zero $(4,0)$ has multiplicity two and it touches the *x*-axis.



The zeros are $(1,0)$ and $(-2,0)$. The zero $(1,0)$ has multiplicity two and it touches the *x*-axis. The zero $(-2,0)$ has multiplicity of one and it crosses the *x*-axis.

NOTE: When the multiplicity of a zero is even, the graph of the function touches the *x*-axis. When the multiplicity of a zero is odd, the graph of the function crosses the *x*-axis.

Practice Exercises E

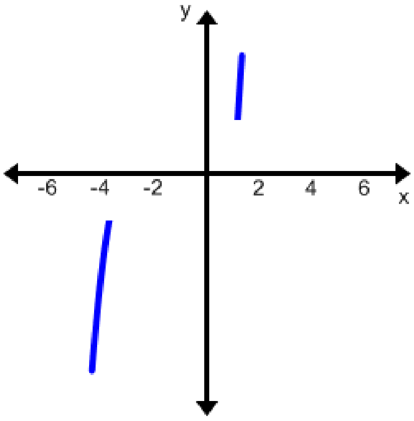
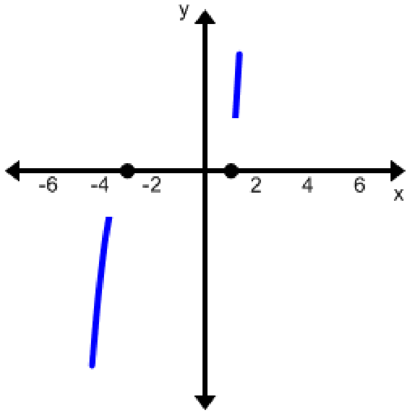
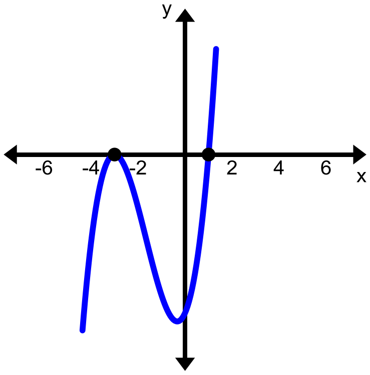
Use technology to graph the polynomial. Identify the zeros, their multiplicity, and determine whether they touch or cross the *x*-axis at each zero.

1. $f(x)=(x+1)^4(x-5)^3$
2. $f(x)=(x-3)(x+2)^2$
3. $f(x)=(x^2-4)(x+5)^3(x-1)^2$
4. $f(x)=(x-1)^3(x^2-9)$
5. $f(x)=(x-2)^2(x+3)^2(x-4)$
6. $f(x)=(x-1)^2(x-4)^3$
7. $f(x)=(x-4)^2(x+1)^3(x+3)$
8. $f(x)=(x-2)^3(x+1)^2(x-5)^2$

Example 8:
Without using technology, sketch each polynomial.

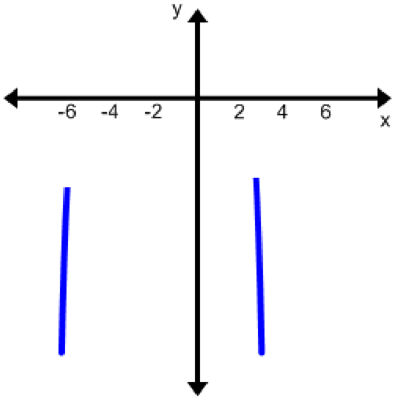
a. $f\left(x\right)=\left(x+3\right)^{2}\left(x-1\right)$

b. $f\left(x\right)=-x\left(x+5\right)^{4}\left(x-2\right)^{3}$

<table>
<tr><td>a. $f\left(x\right)=\left(x+3\right)^{2}\left(x-1\right)$</td><td></td></tr>
<tr><td>End Behavior
 </td><td>$f\left(x\right)$ is a 3rd degree polynomial.
$a_n > 0$ (Leading coefficient is positive)</td></tr>
<tr><td> </td><td>Zeros: $\left(-3,0\right)$ and $\left(1,0\right)$</td></tr>
<tr><td> </td><td><table>
<tr><th>Zero</th><th>Multiplicity</th><th>Touch /Cross</th></tr>
<tr><td>$\left(-3,0\right)$</td><td>2</td><td>Touches</td></tr>
<tr><td>$\left(1,0\right)$</td><td>1</td><td>Crosses</td></tr>
</table></td></tr>
</table>

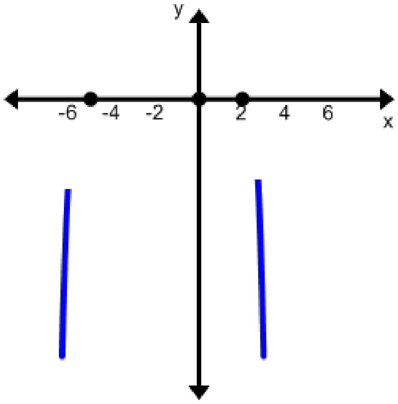
b. $f(x) = -x(x+5)^4(x-2)^3$

End Behavior

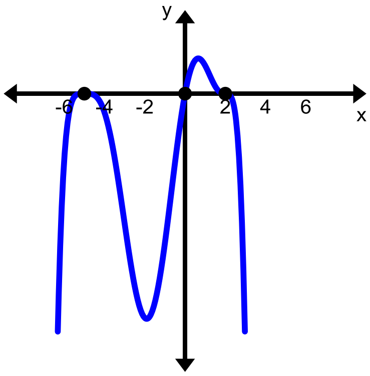


$f(x)$ is an 8th degree polynomial.

$a_n < 0$ (Leading coefficient is negative)



Zeros: $(-5,0), (0,0)$, and $(2,0)$



Zero	Multiplicity	Touch /Cross
$(-5,0)$	4	Touches
$(0,0)$	1	Crosses
$(2,0)$	3	Crosses

Practice Exercises F

Without using technology, sketch each polynomial.

1. $f\left(x\right)=x^3-4x$
2. $f\left(x\right)=\left(x^2-4\right)\left(x^2-1\right)$
3. $f\left(x\right)=\left(x^2-1\right)\left(x^2-9\right)\left(x+2\right)$
4. $f\left(x\right)=\left(x^2-1\right)\left(x^2-9\right)\left(x^2-4\right)$
5. $f\left(x\right)=\left(x-4\right)^2\left(x+2\right)$
6. $f\left(x\right)=\left(x-3\right)^2\left(x+5\right)^2\left(x-1\right)$
7. $f\left(x\right)=\left(x+2\right)^4\left(x-1\right)^5$
8. $f\left(x\right)=\left(x-4\right)^3\left(x+1\right)^2$

Unit 2 Cluster 6: Polynomials
(A.APR.4, A.APR.5, and N.CN.8)

Cluster 6: Polynomial identities
  6.1 Prove polynomial identities
  6.2 Know and apply the binomial theorem
Cluster 1: Use complex numbers in polynomial identities and equations
  1.1 Extend polynomial identities to the complex numbers

Polynomial Identities	
Perfect Square Trinomial $(A+B)^2 = A^2+2AB+B^2$	$(4x+3y)^2 = (4x)^2+2(4x)(3y)+(3y)^2$ $=16x^2+24xy+9y^2$
Difference of Squares $(A+B)(A-B)=A^2-B^2$	$(2x+5y)(2x-5y)=(2x)^2-(5y)^2$ $=4x^2-25y^2$
Cubic Polynomials $(A+B)^3 = A^3+3A^2B+3AB^2+B^3$ $(A-B)^3 = A^3-3A^2B+3AB^2-B^3$	$(2x+5y)^3 = (2x)^3+(3)(2x)^2(5y)+(3)(2x)(5y)^2+(5y)^3$ $=8x^3+60x^2y+150xy^2+125y^3$ $(2x-5y)^3 = (2x)^3-(3)(2x)^2(5y)+(3)(2x)(5y)^2-(5y)^3$ $=8x^3-60x^2y+150xy^2-125y^3$
Sum and Difference of Cubes $A^3+B^3=(A+B)(A^2-AB+B^2)$ $A^3-B^3=(A-B)(A^2+AB+B^2)$	$27x^3+64y^3=(3x+4y)\left[(3x)^2-(3x)(4y)+(4y)^2\right]$ $=(3x+4y)(9x^2-12xy+16y^2)$ $27x^3-64y^3=(3x-4y)\left[(3x)^2+(3x)(4y)+(4y)^2\right]$ $=(3x-4y)(9x^2+12xy+16y^2)$
Trinomial Leading Coefficient 1 $x^2+(a+b)x+ab=(x+a)(x+b)$	$x^2+5x+6=x^2+(2+3)x+(2)(3)$ $=(x+2)(x+3)$ $x^2-5x+6=x^2+(-2-3)x+(-2)(-3)$ $=(x-2)(x-3)$

Quadratic Formula Given $ax^2+bx+c=0$ $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	$2x^2-4x-5=0$ $x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(-5)}}{2(2)}$ $=\frac{4\pm\sqrt{16+40}}{4}$ $=\frac{4\pm\sqrt{56}}{4}$ $=\frac{4\pm2\sqrt{14}}{4}$ $=\frac{2\pm\sqrt{14}}{2}$ $4x^2+9$ $x=\frac{-0\pm\sqrt{0^2-4(4)(9)}}{2(4)}$ $=\frac{\pm\sqrt{-144}}{8}$ $=\frac{\pm12i}{8}$ $=\frac{\pm3i}{2}$
Sum of Squares $A^2+B^2=(A+Bi)(A-Bi)$	$4x^2+9=(2x+3i)(2x-3i)$

VOCABULARY

A polynomial with integer coefficients that cannot be factored into polynomials of lower degree, also with integer coefficients, is called an **irreducible or prime polynomial**.

Practice Exercises A

Multiply the expressions using the polynomial identities.

1. $(x-5)(x+4)$
2. $(2x-12y)^2$
3. $(4x-y)^3$
4. $(2x+3)(4x^2-6x+9)$
5. $(2x-5)^3$
6. $(8x-1)^2$
7. $(9x-8y)(9x+8y)$
8. $(4x-5)(16x^2+20x+25)$
9. $(x-13)(x-3)$
10. $(x+3)^3$
11. $(13x+8i)(13x-8i)$
12. $(12x-13y)(12x+13y)$
13. $(10x+4i)(10x-4i)$
14. $(x+12)(x+2)$
15. $(x-11y)(x^2+11y+121y^2)$
16. $(6x+7)^2$
17. $(4x-11)(4x+11)$
18. $(9x+i)(9x-i)$

Practice Exercises B

Factor the expressions using the polynomial identities.

1. $27x^3 - y^3$
2. $81x^2 - 18xy + y^2$
3. $27x^3 - 54x^2y + 36xy^2 - 8y^3$
4. $4x^2 - 49$
5. $9x^2 + 64$
6. $4x^2 + 52x + 169$
7. $x^2 + 19x + 88$
8. $16x^2 - 100y^2$
9. $343x^3 + 8y^3$
10. $36x^2 + 60x + 25$
11. $x^3 - 15x^2 + 75x - 125$
12. $25x^2 - 121$
13. $144x^2 + 25$
14. $x^3 - 512$
15. $x^2 + 4x - 45$
16. $x^3 + 3x^2y + 3xy^2 + y^3$
17. $x^2 - 9x + 18$
18. $16x^2 + 49$

Use the quadratic formula to solve each equation.

19. $x^2 - 5x - 3 = 0$
20. $-4x^2 + 3x + 1 = 0$
21. $x^2 - x - 2 = 0$
22. $-5x^2 - 2x + 3 = 0$
23. $3x^2 + 7x + 2 = 0$
24. $x^2 + 10x + 11 = 0$

Example 1: Factor $x^2 - 6x + 10$ over the complex numbers.

$x^2 - 6x + 10$	This is an irreducible polynomial. We will have to use the quadratic formula to find the roots for this polynomial.
$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$ $= \dfrac{6 \pm \sqrt{36 - 40}}{2}$ $= \dfrac{6 \pm \sqrt{-4}}{2}$ $= \dfrac{6 \pm 2i}{2}$ $= 3 \pm i$	$a = 1$ $b = -6$ $c = 10$
$\left[x - (3+i)\right]\left[x - (3-i)\right]$	Write as factors $(x - di)(x - fi)$
$(x - 3 - i)(x - 3 + i)$	Simplify

Example 2: Factor x^3-8 over the complex numbers.

x^3-8	This is a difference of cubes. $A^3-B^3=(A-B)(A^2+AB+B^2)$ $A=\sqrt[3]{x^3}=x$ $B=\sqrt[3]{8}=2$
$x^3-8=(x-2)(x^2+2x+4)$	Substitute the values for *A* and *B* into the polynomial identity and simplify.
x^2+2x+4	This is an irreducible polynomial. We will have to use the quadratic formula to find the roots for this polynomial.
$x=\frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}$ $=\frac{-2\pm\sqrt{4-16}}{2}$ $=\frac{-2\pm\sqrt{-12}}{2}$ $=\frac{-2\pm 2i\sqrt{3}}{2}$ $=-1\pm i\sqrt{3}$	$a=1$ $b=2$ $c=4$
$\left[x-\left(-1+i\sqrt{3}\right)\right]\left[x-\left(-1-i\sqrt{3}\right)\right]$	Write as factors $(x-di)(x-fi)$
$\left(x+1-i\sqrt{3}\right)\left(x+1+i\sqrt{3}\right)$	Simplify
$x^3-8=(x-2)(x^2+2x+4)$ $=(x-2)\left(x+1-i\sqrt{3}\right)\left(x+1+i\sqrt{3}\right)$	Write all of the factors as one expression.

Practice Exercises C

Factor each expression over the complex numbers.

1. x^2-4x+5
2. $x^2-2x+10$
3. x^2+4x+8
4. $x^2+8x+17$
5. x^2+4x+7
6. x^2+5
7. x^2+6
8. x^2+8
9. x^3+27
10. x^3-64
11. x^3+1
12. $(4x^2-25)(x^2+9)$
13. $(x^2+5x+6)(x^2+4)$
14. $(4x^2-12x+9)(x^2+3)$
15. $(x^2-2x+3)(x^2+1)$

Binomial Theorem

Exploration 1:

1. Expand $\left(x+1\right)^0$

2. Expand $\left(x+1\right)^1$

3. Expand $\left(x+1\right)^2$

4. Expand $\left(x+1\right)^3$

5. Expand $\left(x+1\right)^4$

6. Write the coefficients of each expansion.

7. Without expanding, determine what the coefficients of the expansion of $\left(x+1\right)^5$.

Answer:

1. $\left(x+1\right)^0 = 1$

2. $\left(x+1\right)^1 = x+1$

3. $\left(x+1\right)^2 = x^2+2x+1$

4. $\left(x+1\right)^3 = x^3+3x^2+3x+1$

5. $\left(x+1\right)^4 = x^4+4x^3+6x^2+4x+1$

6. 1
   1 1
   1 2 1
   1 3 3 1
   1 4 6 4 1

7. 1 5 10 10 5 1

Pascal's Triangle was named in honor of Blaise Pascal. The pattern in the triangle represents the coefficients for a binomial expansion.

Row 0 1
Row 1 1 1
Row 2 1 2 1
Row 3 1 3 3 1
Row 4 1 4 6 4 1
Row 5 1 5 10 10 5 1

Exploration 2
Complete the pattern for Row 6, Row 7, and Row 8.

The Binomial Theorem states that for any positive integer *n*,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n$$

, where $\binom{n}{r} = {}_nC_r$. Although the binomial theorem uses combinations to determine the coefficients for each term, Pascal's triangle can be used to determine the coefficients instead.

Notice the *a* exponents decrease from *n* to 0, while the *b* exponents increase from 0 to *n*.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a^1b^0 + a^0b^1$$
$$(a+b)^2 = a^2b^0 + 2a^1b^1 + a^0b^1$$
$$(a+b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$
$$(a+b)^4 = a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + a^0b^4$$
$$(a+b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$$

Example 3:
Expand $(3x-2y)^5$

$$(3x-2y)^5 : \textit{Use } (a+b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5 \textit{ where } a = 3x \text{ and } b = -2y$$

$$(3x)^5(-2y)^0 + 5(3x)^4(-2y)^1 + 10(3x)^3(-2y)^2 + 10(3x)^2(-2y)^3 + 5(3x)^1(-2y)^4 + (3x)^0(-2y)^5$$
$$243x^5 + 5(81x^4)(-2y) + 10(27x^3)(4y^2) + 10(9x^2)(-8y^3) + 5(3x)(16y^4) + (-32y^5)$$
$$243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$$

Practice Exercises D

Expand each of the binomials using the Binomial Theorem.

1. $(x+3)^6$
2. $(x-2)^7$
3. $(2x-1)^5$
4. $(5x+y)^6$
5. $(4x-3y)^5$
6. $(3x+2y)^4$

Unit 2 Cluster 3 (A.SSE.4): Geometric Series

Cluster 3: Write expressions in equivalent forms to solve problems.

- 2.3 Derive the formula for the sum of a geometric series (when the common ratio is not 1) and use the formula to solve problems.
- H.2.3 Discover and justify the formula for infinite geometric series.
- H.2.3 Discover and justify the formula for a finite arithmetic series.

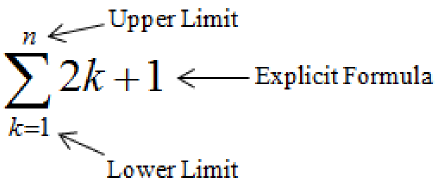
VOCABULARY

A **sequence**, $\{a_1, a_2, a_3, a_4, ...a_n, ...\}$, is a list of terms in a specified order where a_1 is the first term and a_n is the ***n*th term** or the **general term**. A **finite sequence** has a first term and a last term while an **infinite sequence** has a first term, but continues without end.

A **series**, $S = a_1 + a_2 + a_3 + a_4 + ...a_n + ...$, is the sum of the terms of a sequence. A **finite series** sums the terms of a finite sequence while an **infinite series** sums the terms of an infinite sequence.

A **geometric** series is the sum of the terms of a geometric sequence, $S = a + ar + ar^2 + ar^3 + ... + ar^n + ...$, where a is the first term and r is the common ratio.

The Greek capital letter sigma, $\sum$, can be used to indicate the sum of a sequence. The sigma is followed by the explicit formula for the sequence. The small number below the sigma is the first term (lower limit) and the small number above the sigma is the last term (upper limit). The numbers from the lower limit to the upper limit are substituted into the explicit formula and added together.



For example, $\sum_{k=3}^{7} k^3 = 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = 784$.

For an infinite series the upper limit is ∞.

Sigma Notation

Example 1:

Write out and evaluate each sum.

a. $\sum_{k=1}^{5} k^2$ b. $\sum_{k=3}^{8} 2k$ c. $\sum_{k=1}^{4} \frac{k}{k+1}$

a. $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$ $= 1 + 4 + 9 + 16 + 25$ $= 55$	Substitute values into *k* beginning with 1 and ending with 5. Sum each term.

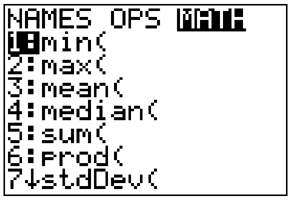
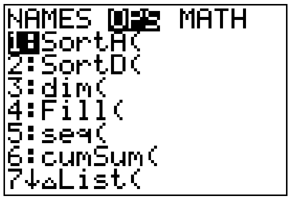
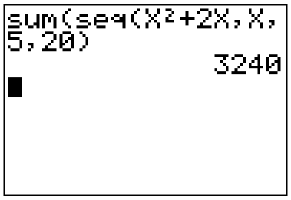
b. $\sum_{k=3}^{8} 2k = (2\cdot 3)+(2\cdot 4)+(2\cdot 5)+(2\cdot 6)+(2\cdot 7)+(2\cdot 8)$ $= 6+8+10+12+14+16$ $= 66$	Substitute values into *k* beginning with 3 and ending with 8. Sum each term.

c. $\sum_{k=1}^{4} \frac{k}{k+1} = \frac{1}{1+1}+\frac{2}{2+1}+\frac{3}{3+1}+\frac{4}{4+1}$ $= \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$ $= \frac{163}{60}$	Substitute values into *k* beginning with 1 and ending with 4. Sum each term.

Example 2:

Use technology to find the sum of $\sum_{k=5}^{20} k^2+2k$.

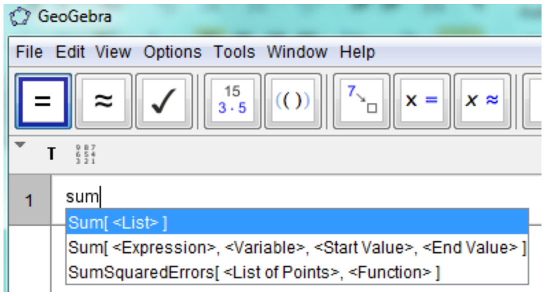
TI-83 or TI-84 Graphing Calculator

Push 2ND LIST STAT then use your arrow keys to arrow over to the MATH menu. Option number 5 is sum(. This will sum all of the terms of the sequence k^2+2k. Select it by pushing L5 5 U or use your arrow keys to arrow down to 5 and then push ENTRY SOLVE ENTER.	
Push 2ND LIST STAT again then use your arrow keys to arrow over to the OPS menu. Option number 5 is seq(. This will allow you to enter the sequence so that its terms can be summed. Select it by pushing L5 5 U or use your arrow keys to arrow down to 5 and then push ENTRY SOLVE ENTER.	
To enter the sequences use the syntax seq(explicit formula, variable, lower limit, upper limit) you can use the variable *x* to enter sum(seq(x^2+2x, *x*, 5, 20). Once you have all of the information entered, push ENTRY SOLVE ENTER and the calculator will find the sum.	 sum(seq(X²+2X,X, 5,20) 3240

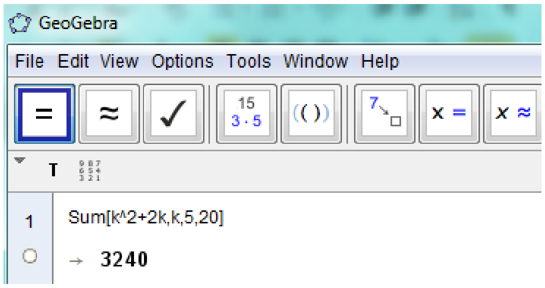
Use technology to find the sum of $\sum_{k=5}^{20} k^2 + 2k$.

Geogebra CAS & Graphics

To bring up the CAS & Graphics window push Ctrl+Shift+k. Begin typing the word sum and options will appear. Select Sum[<Expression>, <Variable>, <Start Value>, <End Value>].



The expression is $k^2 + 2k$, the variable is *k*, the start value is 5, and the end value is 20 (push tab after entering each part). Once all the information is in, push enter and the sum will appear.



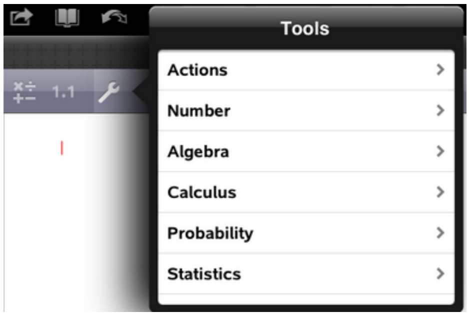
Sum[k^2+2k,k,5,20]

→ 3240

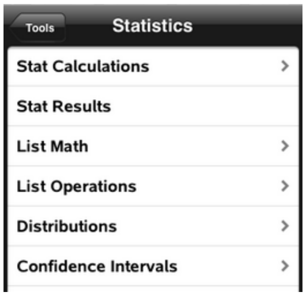
Use technology to find the sum of $\sum_{k=5}^{20} k^2 + 2k$.

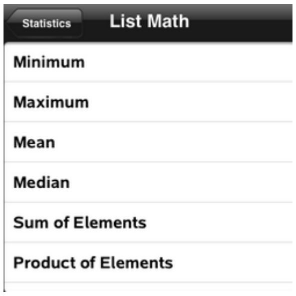
TI-*N*spire CAS on the iPad

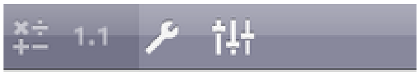
Select a new document by pushing the + at the top left corner. A drop down menu will appear then select Calculator. Bring up the tools menu by pushing the wrench. The tools we are using are under Statistics so select Statistics.



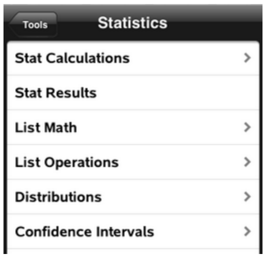
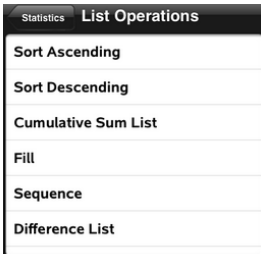
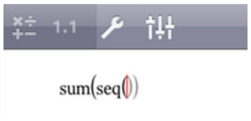
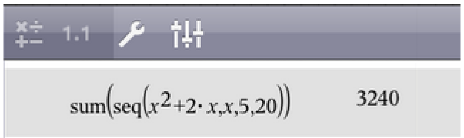
The sum option is under the List Math menu. Select it then select the Sum of Elements. This will sum all the terms of a sequence.







sum()

Bring the tools menu up again by pushing the wrench. Push on Statistics at the top left corner so that the menu goes back to the Statistics menu. Select List Operations. We need to enter a sequence so select Sequence.	   `sum(seq())`
Use the syntax of seq(explicit formula, variable, lower limit, upper limit). Push enter and the sum will appear.	 $\text{sum}(\text{seq}(x^2+2\cdot x,x,5,20))$ 3240

Example 3:
Write each series using sigma notation.

a. $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}...+\frac{1}{10}$ b. $12+18+24+...+54$ c. $50+48+46+...+30$

a. $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}...+\frac{1}{10}$	Determine the explicit formula. In this case it is $\frac{1}{k}$.
$\sum_{k=3}^{10}\frac{1}{k}$	The first term is $\frac{1}{3}$ so the lower limit is 3. The last term is $\frac{1}{10}$ so the upper limit is 10.

b. $12+18+24+...+72$	Determine the explicit formula. This is an arithmetic series. The common difference, d, is 6 and the first term is 12. The explicit formula is $6k+6$ or $6k$, depending on the lower limit.
$\sum_{k=2}^{12}6k$ or $\sum_{k=1}^{11}6k+6$	The first term is 12 which is $6(1)+6$ or $6(2)$ so the lower limit can be 1 or 2. The last term is 72 which is $6(11)+6$ or $6(12)$ so the upper limit can be 11 or 12.

c. $50+48+46+...+30$	Determine the explicit formula. This is an arithmetic series. The common difference, d, is -2 and the first term, a_1, is 50. Use the formula $a_k=a_1+d(k-1)$ to find the explicit formula. $a_k=50+(-2)(k-1)$ $a_k=50-2k+2$ $a_k=52-2k$
$\sum_{k=1}^{11}52-2k$	The first term is 50 which is $52-2(1)$ so the lower limit is 1. The last term is 30 which is $52-2(11)$ so the upper limit is 11.

Practice Exercises A

Write out and evaluate each sum.

1. $\sum_{k=2}^{6} \frac{1}{k^2}$

2. $\sum_{k=1}^{5} 3k-2$

3. $\sum_{k=1}^{5} 2^k - 1$

4. $\sum_{k=1}^{4} (-1)^k k$

5. $\sum_{k=3}^{7} \frac{k}{k+2}$

6. $\sum_{k=3}^{10} 2k+1$

Write each series using sigma notation.

7. $5+7+9+...+17$

8. $6+5+4+...+(-1)$

9. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...+\frac{1}{20}$

10. $1+4+9+16+...+49$

11. $37+34+31+...+13$

12. $\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+...+\frac{9}{64}$

Deriving the Formula for the Sum of a Finite Geometric Series

Exploration:

1. Multiply: $(x-1)(x+1)$

2. Multiply: $(x-1)(x^2+x+1)$

3. Multiply: $(x-1)(x^3+x^2+x+1)$

4. Multiply: $(x-1)(x^4+x^3+x^2+x+1)$

5. Without multiplying, determine the product of $(x-1)(x^{n-1}+x^{n-2}+...+x^2+x+1)$.

Answer:

1. $(x-1)(x+1)=x^2-1$
2. $(x-1)(x^2+x+1)=x^3-1$
3. $(x-1)(x^3+x^2+x+1)=x^4-1$
4. $(x-1)(x^4+x^3+x^2+x+1)=x^5-1$
5. $(x-1)(x^{n-1}+x^{n-2}+...+x^2+x+1)=x^n-1$

In general, a finite geometric series has the form $\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$. If you rewrite the sum of a geometric series in standard polynomial form (with the exponents in descending order), then you get: $ar^{n-1} + ... + ar^3 + ar^2 + ar + a$. Factoring out the first term you get: $a\left(r^{n-1} + ... + r^3 + r^2 + r + 1\right)$. Which looks a lot like $x^{n-1} + x^{n-2} + ... + x^2 + x + 1$. We know that $(x-1)\left(x^{n-1} + x^{n-2} + ... + x^2 + x + 1\right) = x^n - 1$. If we solve the equation for $x^{n-1} + x^{n-2} + ... + x^2 + x + 1$, then we get $x^{n-1} + x^{n-2} + ... + x^2 + x + 1 = \frac{x^n - 1}{x - 1}$. With a little bit of algebra we get $\frac{x^n - 1}{x - 1} = \frac{(-1)(1 - x^n)}{(-1)(1 - x)} = \frac{1 - x^n}{1 - x}$. Since $r^{n-1} + ... + r^3 + r^2 + r + 1$ looks like $x^{n-1} + x^{n-2} + ... + x^2 + x + 1$ we can conclude that $a\left(r^{n-1} + ... + r^3 + r^2 + r + 1\right) = a\left(\frac{1 - r^n}{1 - r}\right)$.

Sum of a Finite Geometric Series

The formula for the sum of a finite geometric series is $S_n = \frac{a(1 - r^n)}{1 - r}$ where *a* is the first term, *r* is the common ratio, and *n* is the number of terms.

Example 4:
Evaluate the sum of the finite geometric series.

a. $1 + 3 + 9 + 27 + 81 + 243 + 729$

b. $\sum_{k=1}^{10} 5\left(\frac{1}{2}\right)^{k-1}$

a. $1 + 3 + 9 + 27 + 81 + 243 + 729$	Use $S_n = \frac{a(1 - r^n)}{1 - r}$.
$S_7 = \frac{1(1 - 3^7)}{1 - 3}$ $S_7 = \frac{-2186}{-2}$ $S_7 = 1093$	The common ratio is 3. $r = 3$ The first term is 1. $a = 1$ There are 7 terms. $n = 7$

b. $\sum_{k=1}^{10}\left(\frac{1}{2}\right)^{k-1}$	Use $S_n=\frac{a\left(1-r^n\right)}{1-r}$.
$S_{10}=\frac{1\left(1-\frac{1^{10}}{2}\right)}{1-\frac{1}{2}}$ $S_{10}=\frac{\frac{1023}{1024}}{\frac{1}{2}}$ $S_{10}=\frac{2046}{1024}\approx 1.998$	The common ratio is $\frac{1}{2}$. $r=\frac{1}{2}$ The first term is $\left(\frac{1}{2}\right)^{1-1}=1$. $a=1$ The upper limit is 10. $n=10$

Practice Exercises B

Evaluate the sum of the finite geometric series.

1. $1+4+16+...+4096$
2. $3+6+12+24+...+768$
3. $4-12+36-...-8748$
4. $81+27+...+\frac{1}{9}$
5. $1+\frac{1}{5}+\frac{1}{25}+...+\frac{1}{3125}$
6. $\frac{7}{10}-\frac{7}{100}+...+\frac{7}{100,000}$
7. $\sum_{k=1}^{5}(-2)(-3)^{k-1}$
8. $\sum_{k=1}^{6}(-1)(-5)^{k-1}$
9. $\sum_{k=1}^{5}(-2)(6)^{k-1}$
10. $\sum_{k=1}^{7}(-3)\left(\frac{1}{4}\right)^{k-1}$
11. $\sum_{k=1}^{6}3\left(\frac{1}{2}\right)^{k-1}$
12. $\sum_{k=1}^{7}4\left(\frac{2}{3}\right)^{k-1}$
13. A professional baseball player signs a contract with a beginning salary of $2,250,000 for the first year and an annual increase of 5% per year beginning in the second year. How much money in total will the athlete make if his contract is for 6 years? Round to the nearest dollar.
14. You are investigating two employment opportunities. Company A offers $33,000 the first year. During the next four years the salary is guaranteed to increase by 7% per year. Company B offers $35,000 the first year, with guaranteed annual increases of 4% per year after that. Which company offers the better total salary for a five-year contract?
15. A job starts at $28,700 with a yearly increase of 2.5% after the first year. The average person retires after 30 years. What is the total life-time salary over the 30-year period? Round to the nearest dollar.

Sum of an Infinite Geometric Series (Honors)

Calculus is required to derive the formula for the sum of an infinite geometric series. Informally, we can prove that $S=\dfrac{a}{1-r}$ if $|r|<1$ by looking at an example. The repeating decimal $0.\bar{3}$ can be written as a geometric series as follows $\sum_{k=1}^{n} 3\left(0.1\right)^{k}$. The first term, *a*, would be $3\left(0.1\right)^{1}=0.3$ and the common ratio, *r*, 0.1.

n	r^n	$a\left(1-r^n\right)$	$\dfrac{a\left(1-r^n\right)}{1-r}$
1	0.1	$0.3\left(1-\left(0.1\right)^{1}\right)=0.27$	$\dfrac{0.27}{1-0.1}=.3$
2	0.01	$0.3\left(1-\left(0.1\right)^{2}\right)=0.297$	$\dfrac{0.297}{1-0.1}=.33$
3	0.001	$0.3\left(1-\left(0.1\right)^{3}\right)=0.2997$	$\dfrac{0.2997}{1-0.1}=.333$
4	0.0001	$0.3\left(1-\left(0.1\right)^{4}\right)=0.29997$	$\dfrac{0.29997}{1-0.1}=.3333$
5	0.00001	$0.3\left(1-\left(0.1\right)^{5}\right)=0.299997$	$\dfrac{0.299997}{1-0.1}=.33333$
6	0.000001	$0.3\left(1-\left(0.1\right)^{6}\right)=0.2999997$	$\dfrac{0.2999997}{1-0.1}=.333333$
7	0.0000001	$0.3\left(1-\left(0.1\right)^{7}\right)=0.29999997$	$\dfrac{0.29999997}{1-0.1}=.3333333$
8	0.00000001	$0.3\left(1-\left(0.1\right)^{8}\right)=0.299999997$	$\dfrac{0.299999997}{1-0.1}=.33333333$
9	0.000000001	$0.3\left(1-\left(0.1\right)^{9}\right)=0.2999999997$	$\dfrac{0.2999999997}{1-0.1}=.333333333$
10	0.0000000001	$0.3\left(1-\left(0.1\right)^{10}\right)=0.29999999997$	$\dfrac{0.29999999997}{1-0.1}=.3333333333$

By looking at the values in the table you can see that as *n* gets infinitely larger r^n gets closer to zero, which makes $a\left(1-r^n\right)$ approach the value of *a*. Thus, the sum of an infinite geometric series for $|r|<1$ is $S=\dfrac{a}{1-r}$. When $|r|<1$ the geometric series has a finite sum and it is said that the series **converges.**

Through a similar process we can show that for $|r| \geq 1$ there is no finite sum. Consider the geometric series $\sum_{k=1}^{n} 2(1.01)^k$. The first term, *a*, is $2(1.01)^1 = 2.02$ and the common ratio, *r*, is 1.01.

n	r^n	$a(1-r^n)$	$\frac{a(1-r^n)}{1-r}$
1	1.01	$2.02(1-(1.01)^1) = -0.0202$	$\frac{-0.0202}{1-1.01} = 2.02$
2	1.0201	$2.02(1-(1.01)^2) = -0.0406$	$\frac{-0.0406}{1-1.01} = 4.0602$
3	1.0303	$2.02(1-(1.01)^3) = -0.0612$	$\frac{-0.0612}{1-1.01} = 6.120802$
4	1.0406	$2.02(1-(1.01)^4) = -0.082$	$\frac{-0.082}{1-1.01} = 8.20201002$
5	1.051	$2.02(1-(1.01)^5) = -0.103$	$\frac{-0.103}{1-1.01} = 10.3040301$
10	1.104622125	$2.02(1-(1.01)^{10}) \approx -0.2113367$	$\frac{-0.2113367}{1-1.01} \approx 21.13367$
100	2.704814	$2.02(1-(1.01)^{100}) \approx -3.443724$	$\frac{-3.443724}{1-1.01} \approx 344.3723955$

You can see that as *n* gets larger both r^n and $a(1-r^n)$ get larger, which makes the sum $S = \frac{a(1-r^n)}{1-r}$ grow without bound. Thus, the sum of an infinite geometric series for $|r| \geq 1$ is not a finite number. It is said that a geometric series with $|r| \geq 1$ **diverges** because there is no finite sum.

Sum of an Infinite Geometric Series

The formula for the sum of an infinite geometric series is $S = \frac{a}{1-r}$ where *a* is the first term and *r*, such that $|r| < 1$, is the common ratio. There is no finite sum for an infinite geometric series if $|r| \geq 1$.

Example 5:
Identify the common ratio. Then determine if the geometric series will converge or diverge.

a. $\frac{5}{6} + \frac{25}{36} + \frac{125}{216} + \ldots$

b. $\sum_{k=1}^{\infty} \left(\frac{4}{\pi}\right)^k$

a. $\frac{5}{6}+\frac{25}{36}+\frac{125}{216}+...$ The common ratio is $\frac{25}{36}\div\frac{5}{6}=\frac{25}{36}\cdot\frac{6}{5}=\frac{5}{6}$. Since $\left\lvert\frac{5}{6}\right\rvert<1$, the geometric series will converge.	b. $\sum_{k=1}^{\infty}\left(\frac{4}{\pi}\right)^{k}$ The common ratio is $\frac{4}{\pi}\approx 1.273$. Since $\left\lvert\frac{4}{\pi}\right\rvert\geq 1$, the geometric series will diverge.

Example 6:
Find the sum of the infinite geometric series.

a. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+...$

b. $\sum_{k=1}^{\infty}5\left(\frac{1}{\pi}\right)^{k}$

a. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+...$	
$S=\frac{\frac{1}{2}}{1-\left(-\frac{1}{2}\right)}=\frac{\frac{1}{2}}{\frac{2}{2}+\frac{1}{2}}$ $S=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$	The common ratio is $-\frac{1}{4}\div\frac{1}{2}=-\frac{1}{4}\cdot 2=-\frac{1}{2}$ and $\left\lvert-\frac{1}{2}\right\rvert<1$. The first term is $\frac{1}{2}$. Use the formula $S=\frac{a}{1-r}$.

b. $\sum_{k=1}^{\infty}5\left(\frac{1}{\pi}\right)^{k}$	
$S=\frac{\frac{5}{\pi}}{1-\frac{1}{\pi}}=\frac{\frac{5}{\pi}}{\frac{\pi}{\pi}-\frac{1}{\pi}}$ $S=\frac{\frac{5}{\pi}}{\frac{\pi-1}{\pi}}=\frac{5}{\pi}\cdot\frac{\pi}{\pi-1}=\frac{5}{\pi-1}$	The common ratio is $\frac{1}{\pi}$ and $\left\lvert\frac{1}{\pi}\right\rvert<1$. The first term is $5\left(\frac{1}{\pi}\right)^{1}=\frac{5}{\pi}$. Use the formula $S=\frac{a}{1-r}$.

Practice Exercises C

Identify the common ratio. Then determine if the geometric series will converge or diverge.

1. $0.0004-0.004+0.04-...$

2. $-\frac{1}{24}+\frac{1}{12}-\frac{1}{6}+\frac{1}{3}+...$

3. $3+\frac{3}{4}+\frac{3}{4^2}+\frac{3}{4^3}+...$

4. $\sum_{k=1}^{\infty} 47\left(0.01\right)^k$

5. $\sum_{k=1}^{\infty} \frac{1}{2}\left(1.02\right)^{k-1}$

6. $\sum_{k=1}^{\infty} 18\left(\frac{1}{3}\right)^{k-1}$

Find the sum of each infinite geometric series.

7. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...$

8. $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+...$

9. $2-\frac{2}{5}+\frac{2}{25}-\frac{2}{125}+...$

10. $1024+128+16+2+...$

11. $12+6+3+\frac{3}{2}+...$

12. $1+\frac{e}{\pi}+\frac{e^2}{\pi^2}+\frac{e^3}{\pi^3}+...$

13. $\sum_{k=1}^{\infty} 3\left(0.1\right)^{k-1}$

14. $\sum_{k=1}^{\infty} \left(\frac{2}{\pi}\right)^k$

15. $\sum_{k=1}^{\infty} \left(\frac{e}{3}\right)^{k-1}$

16. $\sum_{k=1}^{\infty} 83\left(\frac{1}{100}\right)^k$

17. $\sum_{k=1}^{\infty} -2\left(0.6\right)^{k-1}$

18. $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{k-1}$

19. The height a ball bounces is less than the height of the previous bounce due to friction. Suppose a ball is dropped from a height of 4 feet and rebounds to 98% of the height of the previous bounce. Write the series in sigma notation. What is the total vertical distance traveled by the ball when it comes to rest?

20. Because of friction and air resistance, each swing of a pendulum is a little shorter than the previous one. Suppose the first swing of a pendulum has a length of 5 inches and the return swing is 4.8 inches. Write the series in sigma notation. What is the total distance traveled by the pendulum when it comes to rest?

Deriving the Sum of a Finite Arithmetic Series (Honors)

VOCABULARY

An **arithmetic series** is the sum of the terms of an arithmetic sequence. An arithmetic sequence can be written explicitly using the formula $a_n = a_1 + (n-1)d$, where a_1 is the first term and *d* is the common difference. It can also be written recursively using the formula $a_n = a_{n-1} + d$, where a_{n-1} is the previous term and *d* is the common difference.

The sum of the first *n* terms of an arithmetic sequence would be: $S_n = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$. Using the explicit formula for the *n*th term the sum can be rewritten as $S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_1 + (n-2)d) + (a_1 + (n-1)d)$. The sum of the first *n* terms could also be written as $S_n = a_n + a_{n-1} + ... + a_3 + a_2 + a_1$. Using the explicit formula for the *n*th term the sum can be rewritten as $S_n = a_n + (a_n - d) + (a_n - 2d) + ... + (a_n - (n-2)d) + (a_n - (n-1)d)$. If you were to add the two equations together then you would get:

$$\begin{array}{l} S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_1 + (n-2)d) + (a_1 + (n-1)d) \\ \underline{+S_n = a_n + (a_n - d) + (a_n - 2d) + ... + (a_n - (n-2)d) + (a_n - (n-1)d)} \\ 2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + ... + (a_1 + a_n) + (a_1 + a_n) \end{array}$$

This result can be simplified to $2S_n = n(a_1 + a_n)$ since there are *n* sums of $a_1 + a_n$. Solving the equation for S_n produces $S_n = \frac{n}{2}(a_1 + a_n)$. If you were to substitute the explicit formula in for a_n then the result would be $S_n = \frac{n}{2}\left[a_1 + a_1 + (n-1)d\right] = \frac{n}{2}\left[2a_1 + (n-1)d\right]$. This can also be used to find the sum of an arithmetic series.

The Sum of a Finite Arithmetic Series

The formula for the sum of a finite arithmetic series is $S_n = \frac{n}{2}(a_1 + a_n)$, where a_1 is the first term, a_n is the last term, and *n* is the number of terms.

The sum of a finite arithmetic series can also be found using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, where a_1 is the first term, *d* is the common difference and *n* is the number of terms.

Example 7:
Find the sum of the finite arithmetic series.

a. $-5-11-17-23-\ldots-71$

b. $\sum_{k=1}^{8} 4k+3$

a. $-5-11-17-23-\ldots-71$	
$S_{12}=\frac{12}{2}(-5+-71)$ $S_{12}=6(-76)$ $S_{12}=-456$	Use the formula $S_n=\frac{n}{2}(a_1+a_n)$. The common difference is $-11-(-5)=-6$. The first term is -5. The last term is -71. The number of terms can be found using the explicit formula $a_n=a_1+(n-1)d$. $-71=-5+(n-1)(-6)$ $-71=-5-6n+6$ $-71=1-6n$ $-72=-6n$ $12=n$

b. $\sum_{k=1}^{8} 4k+3$	
$S_8=\frac{8}{2}(7+35)$ $S_8=4(42)$ $S_8=168$	Use the formula $S_n=\frac{n}{2}(a_1+a_n)$. The common difference is 4. The first term is $4(1)+3=7$. The last term is $4(8)+3=35$. The number of terms is 8.

Practice Exercises D

Find the sum of the finite arithmetic series.

1. $8+13+18+23+...+88$
2. $2+8+14+20+...+116$
3. $2+(-1)+(-4)+...+(-40)$
4. $4+2+0+....+(-20)$
5. $7+19+31+43+...+115$
6. $1+5+9+13+...+45$
7. $\sum_{k=1}^{25}(4k-14)$
8. $\sum_{k=1}^{20}(6k-21)$
9. $\sum_{k=1}^{15}(11-5k)$
10. $\sum_{k=1}^{17}(10-4k)$
11. $\sum_{k=1}^{14}(3k-1)$
12. $\sum_{k=3}^{22}(2k-1)$
13. A theater has 20 seats in the first row, 22 seats in the second row, increasing by 2 seats per row for a total of 25 rows.

    a. Write an arithmetic series to represent the number of seats in the theater.

    b. Find the total seating capacity of the theater.

    c. If tickets are \$9.25 per seat, how much money will the theater make if the theater is filled to capacity?

14. A supermarket displays cans in a triangle. There are 15 cans in the bottom row and each successive row has one fewer can than the previous row for a total of 14 rows.

    a. Use summation notation to write the series for the triangle.

    b. How many cans are in the display?

15. A company offers a starting yearly salary of \$28,500 with raises of \$1,000 each year after the first year. Find the total salary over a 15-year period.

Unit 2 Cluster 7 (A.APR.6 and A.APR.7): Rational Expressions

Cluster 7: Rewrite rational expressions

2.7	Rewrite simple rational expressions in different forms using inspection, long division, or, for the more complicated examples, a computer algebra system (CAS).
2.7	Add, subtract, multiply, and divide rational expressions.
2.7	Closure of rational expressions under addition, subtraction, multiplication, and division by a nonzero rational expression.

VOCABULARY

A **rational function** is a function of the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x)\neq 0$. For example, $f(x)=\frac{3x-4}{x+1}$ is a rational function.

Example 1:

Simplify $f(x)=\frac{-5x^3+4x^2+6x}{x}$

$f(x)=\frac{-5x^3+4x^2+6x}{x}$	$x\neq 0$ because the denominator must be a nonzero polynomial
$f(x)=\frac{-5x^3}{x}+\frac{4x^2}{x}+\frac{6x}{x}$	Rewrite the rational expression as the sum of fractions with a common denominator.
$f(x)=-5x^2+4x+6$	Simplify

Example 2:

Simplify $f(x)=\frac{x^2-4}{x+2}$

$f(x)=\frac{x^2-4}{x+2}$	$x\neq -2$ because the denominator must be a nonzero polynomial
$f(x)=\frac{(x+2)(x-2)}{x+2}$	Factor
$f(x)=\frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$	Simplify like terms
$f(x)=x-2$	

Example 3:

Simplify $f(x)=\dfrac{x^2-5x+4}{x^2+2x-3}$

$f(x)=\dfrac{x^2-5x+4}{x^2+2x-3}$	
$f(x)=\dfrac{(x-4)(x-1)}{(x+3)(x-1)}$	Factor $x\neq-3$ or $x\neq1$ because the denominator must be a nonzero polynomial
$f(x)=\dfrac{(x-4)\cancel{(x-1)}}{(x+3)\cancel{(x-1)}}$	Simplify like terms
$f(x)=\dfrac{(x-4)}{(x+3)}$	

Example 4:

Simplify $f(x)=\dfrac{x^2-8x+15}{x^3-125}$

$f(x)=\dfrac{x^2-8x+15}{x^3-125}$	
$f(x)=\dfrac{(x-3)(x-5)}{(x-5)(x^2+5x+25)}$	Factor $x\neq5$ because the denominator must be a nonzero polynomial
$f(x)=\dfrac{(x-3)\cancel{(x-5)}}{\cancel{(x-5)}(x^2+5x+25)}$	Simplify like terms
$f(x)=\dfrac{(x-3)}{(x^2+5x+25)}$	

Practice Exercises A

Simplify each rational expression.

1. $\dfrac{8x^3-4x^2-2x}{2x}$
2. $\dfrac{-4x^3-10x^2+2x}{2}$
3. $\dfrac{3x^2+4x+9}{x}$
4. $\dfrac{x^2-8x-20}{x-10}$
5. $\dfrac{x^2-4x-5}{x+1}$
6. $\dfrac{x-9}{x^2-18x+81}$
7. $\dfrac{6x^2-47x-8}{x-8}$
8. $\dfrac{3x^2+25x+42}{3x+7}$
9. $\dfrac{2x+7}{4x^2-49}$

10. $\dfrac{x^2-4}{x^2+4x+4}$

11. $\dfrac{x^2+16x+63}{x^2+3x-54}$

12. $\dfrac{x^2+9x+8}{x^2+16x+64}$

13. $\dfrac{2x^2-13x-7}{2x^2+21x+10}$

14. $\dfrac{25x^2-4}{5x^2+8x-4}$

15. $\dfrac{3x^2+25x-18}{x^2+10x+9}$

16. $\dfrac{x^3-1}{x-1}$

17. $\dfrac{x+2}{x^3+8}$

18. $\dfrac{x^3+216}{x+6}$

19. $\dfrac{x^2-16}{x^3+64}$

20. $\dfrac{27x^3-8}{3x^2+16x-12}$

21. $\dfrac{4x^2-2x+1}{8x^3+1}$

Long Division with Polynomials

A rational expression $f(x)=\dfrac{p(x)}{q(x)}$ can be thought of as $p(x)$ divided by $q(x)$. Division of polynomials, like division of real numbers, uses multiplication and subtraction.

Let us review the algorithm for long division with real numbers. Consider 335 divided by 23.

$335 \div 23$	335 is the dividend 23 is the divisor
$23\overline{)335}$	Rewrite the division problem so that the dividend is under the long division symbol and the divisor is on the outside
$\begin{array}{r} 1 \\ 23\overline{)\ 335} \\ \underline{-23} \\ 105 \end{array}$	Divide 33 by 23. $\dfrac{33}{23}$ is a little over one. This becomes the first term of your quotient. Multiply 23 by 1 and subtract the product from the dividend. Bring down the next unused term. This is your new dividend.
$\begin{array}{r} 14 \\ 23\overline{)\ 335} \\ \underline{-23} \\ 105 \\ \underline{-92} \\ 13 \end{array}$	Divide 105 by 23. $\dfrac{105}{23}$ is a little over four. This becomes the next term of your quotient. Multiply 23 by 4 and subtract the product from the dividend. The remainder is 13. Therefore, $335 \div 23 = 14 + \dfrac{13}{23}$

Example 5:

Simplify using long division $\frac{2x^2-5x-12}{x-4}$

$\frac{2x^2-5x-12}{x-4}$	$2x^2-5x-12$ is the dividend $x-4$ is the divisor
$x-4\overline{)2x^2-5x-12}$	Rewrite the rational expression with the dividend under the long division symbol and the divisor on the outside.
$\begin{array}{r} 2x \\ x-4\overline{)\quad 2x^2-5x-12} \\ \underline{-\left(2x^2-8x\right)} \\ 3x-12 \end{array}$	Divide $2x^2$ by x. $\frac{2x^2}{x}=2x$ this becomes the first term of your quotient. Multiply $x-4$ by $2x$ and subtract the product from the dividend. Bring down the next unused term. This is your new dividend**.**
$\begin{array}{r} 2x+3 \\ x-4\overline{)\quad 2x^2-5x-12} \\ \underline{-\left(2x^2-8x\right)} \\ 3x-12 \\ \underline{-\left(3x-12\right)} \\ 0 \end{array}$	Divide $3x$ by x. $\frac{3x}{x}=3$ this becomes the next term of your quotient. Multiply $x-4$ by 3 and subtract the product from the dividend. The remainder is zero. Therefore, $\frac{2x^2-5x-12}{x-4}=2x+3$

Example 6:

Simplify using long division $\frac{3x^3+5x^2+8x+7}{3x+2}$

$\frac{3x^3+5x^2+8x+7}{3x+2}$	$3x^3+5x^2+8x+7$ is the dividend $3x+2$ is the divisor
$3x+2\overline{)3x^3+5x^2+8x+7}$	Rewrite the rational expression with the dividend under the long division symbol and the divisor on the outside.
$\begin{array}{r} x^2 \\ 3x+2\overline{)\quad 3x^3+5x^2+8x+7} \\ \underline{-\left(3x^3+2x^2\right)} \\ 3x^2+8x \end{array}$	Divide $3x^3$ by $3x$. $\frac{3x^3}{3x}=x^2$ this becomes the first term of your quotient. Multiply $3x+2$ by x^2 and subtract the product from the dividend. Bring down the next unused term. This is your new dividend**.**

$\begin{array}{r} x^2 + x \\ 3x+2\overline{)\,3x^3+5x^2+8x+7} \\ -\left(3x^3+2x^2\right) \\ \hline 3x^2+8x \\ -\left(3x^2-2x\right) \\ \hline 6x+7 \end{array}$	Divide $3x^2$ by $3x$. $\frac{3x^2}{3x}=x$ this becomes the next term of your quotient. Multiply $3x+2$ by x and subtract the product from the dividend. Bring down the next unused term. This is your new dividend.
$\begin{array}{r} x^2 + x + 2 \\ 3x+2\overline{)\,3x^3+5x^2+8x+7} \\ -\left(3x^3+2x^2\right) \\ \hline 3x^2+8x \\ -\left(3x^2-2x\right) \\ \hline 6x+7 \\ -\left(6x+4\right) \\ \hline 3 \end{array}$	Divide $6x$ by $3x$. $\frac{6x}{3x}=2$ this becomes the next term of your quotient. Multiply $3x+2$ by 2 and subtract the product from the dividend. The remainder is 3. Therefore, $\frac{3x^3+5x^2+8x+7}{3x+2}=x^2+x+2+\frac{3}{3x+2}$

Practice Exercises B

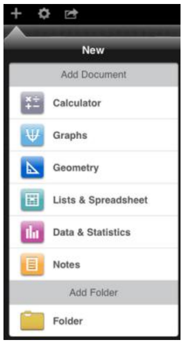
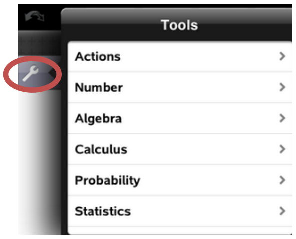
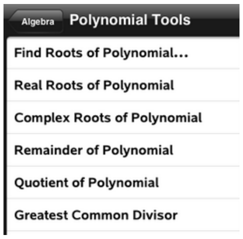
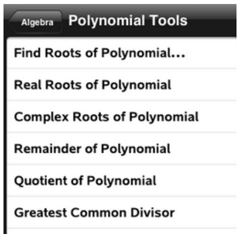
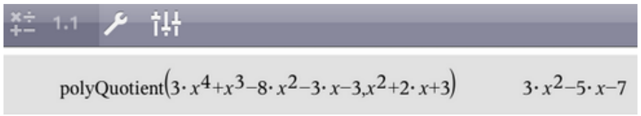
Simplify using long division.

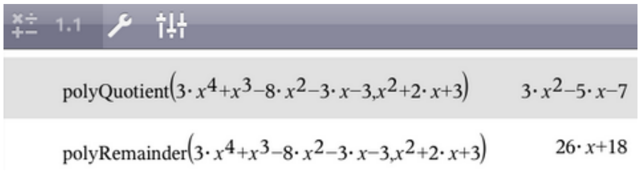
1. $\frac{x^2-10x-25}{x-5}$
2. $\frac{x^2-8x-16}{x+4}$
3. $\frac{x^2-9x+21}{x-4}$
4. $\frac{3x^2-11x+33}{x-6}$
5. $\frac{4x^2-2x+3}{x-1}$
6. $\frac{10x^2-6x+3}{5x+2}$
7. $\frac{6x^2-7x+3}{3x-2}$
8. $\frac{x^3-2x^2+4x-5}{x+3}$
9. $\frac{x^3-1}{x+1}$
10. $\frac{x^3+4x^2+7x-9}{x+3}$
11. $\frac{2x^3+3x^2-x-3}{x+2}$
12. $\frac{10x^3+6x^2-9x+10}{5x-2}$
13. $\frac{6x^3-11x^2+11x-2}{2x-3}$
14. $\frac{4x^3-8x^2+3x-1}{2x+1}$
15. $\frac{3x^3-5x^2-3x-2}{x-2}$

Using Technology to Divide Polynomials

For more complicated polynomial division you may want to use a computer algebra system such as the TI-Nspire CAS. You can download an app of the TI-Nspire CAS for your iPad or you can purchase a TI-Nspire CAS calculator. The instructions below are for the iPad app version.

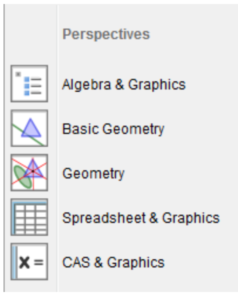
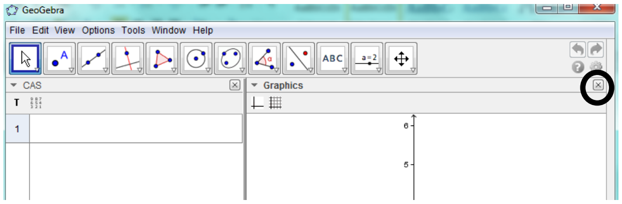
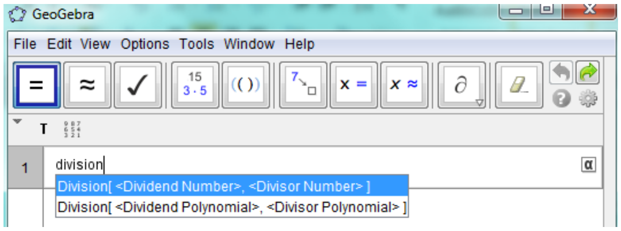
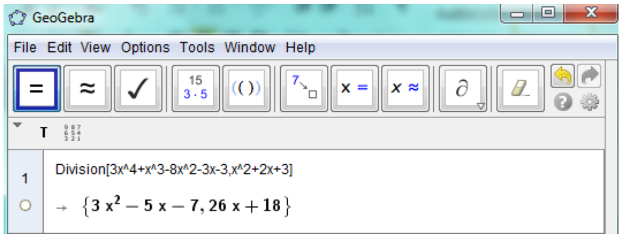
Example 7: Divide $3x^4+x^3-8x^2-3x-3$ by x^2+2x+3

Create a new document by pushing the + symbol in the upper left hand corner. A menu, like the one at the right, will appear. Select Calculator.	
You will have a document that you can type mathematical equations in.	
Push the wrench at the top of the screen to bring up the Tools menu then select Algebra. A new Algebra menu will appear. Scroll down and Select Polynomial Tools.	 
The Polynomial Tools menu should have Remainder of Polynomial and Quotient of Polynomial. You will need both of these tools to divide.	
Select Quotient of Polynomial. Enter the dividend polynomial (for this example it is $3x^4+x^3-8x^2-3x-3$) then a comma and the divisor polynomial (for this example it is x^2+2x+3). Press Enter and you will have the quotient, but not the remainder if there is one.	 $\text{polyQuotient}(3\cdot x^4+x^3-8\cdot x^2-3\cdot x-3,x^2+2\cdot x+3)$ $3\cdot x^2-5\cdot x-7$

Push the wrench to bring up the Polynomial Tools menu again. Select Remainder of Polynomial. Enter the dividend polynomial (for this example it is $3x^4+x^3-8x^2-3x-3$) then a comma and the divisor polynomial (for this example it is x^2+2x+3). Press Enter and you will have the remainder.	 polyQuotient$(3\cdot x^4+x^3-8\cdot x^2-3\cdot x-3,x^2+2\cdot x+3)$ $3\cdot x^2-5\cdot x-7$ polyRemainder$(3\cdot x^4+x^3-8\cdot x^2-3\cdot x-3,x^2+2\cdot x+3)$ $26\cdot x+18$
$3x^4+x^3-8x^2-3x-3$ divided by x^2+2x+3 is $3x^2-5x-7+\dfrac{26x+18}{x^2+2x+3}$.	

You can also use Geogebra as a computer algebra system. The program is free and can be downloaded to your computer at www.geogebra.org. (Instructions for PC version.)

Example 7B: Divide $3x^4+x^3-8x^2-3x-3$ by x^2+2x+3

When you open Geogebra you should see a menu like the one at the right. Select CAS & Graphics. If it there is no menu, then press ctrl + shift + k to bring up the CAS screen.	
You should see a screen like the one at the right. To close the Graphics screen click on the x in the right hand corner.	
To the right of the number 1, start typing division and some options will come up. Select Division [<Dividend Polynomial>, <Divisor Polynomial>].	 1 division Division[<Dividend Number>, <Divisor Number>] Division[<Dividend Polynomial>, <Divisor Polynomial>]
The dividend polynomial is $3x^4+x^3-8x^2-3x-3$ and the divisor polynomial is x^2+2x+3. Once you have entered the polynomials, press enter and you will get an answer in the form {Quotient,Remainder}.	 1 Division[3x^4+x^3-8x^2-3x-3,x^2+2x+3] → $\{3x^2-5x-7, 26x+18\}$
$3x^4+x^3-8x^2-3x-3$ divided by x^2+2x+3 is $3x^2-5x-7+\dfrac{26x+18}{x^2+2x+3}$.	

Practice Exercises C

Use a computer algebra system to divide the polynomials.

1. $\dfrac{-5x^4 - x^3 + 31x^2 - 31x - 6}{-5x^2 + 9x - 2}$
2. $\dfrac{3x^4 - x^3 - 8x^2 + 5x - 4}{-3x^2 + x - 7}$
3. $\dfrac{x^4 + 8x^3 + 17x^2 + 6x - 13}{x^2 + 5x + 2}$
4. $\dfrac{-2x^4 - 20x^3 + 6x^2 + 20x + 3}{x^2 + 10x - 2}$
5. $\dfrac{8x^4 - 2x^3 - 18x^2 - 5}{4x^2 - x - 9}$
6. $\dfrac{-7x^5 - 3x^3 + 63x^2 + 7}{x^3 - 9}$

Multiplying and Dividing Rational Expressions

VOCABULARY

If the numerator and denominator of a rational expression have no common factors, other than ± 1, then the rational expression is in **simplified form**.

To multiply a rational expression by another, multiply the numerator with the numerator and the denominator with the denominator.

$\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd} \longleftarrow$ Simplify $\dfrac{ac}{bd}$ if possible ($b \neq 0$ and $d \neq 0$)

To divide one rational expression by another, multiply the first expression by the reciprocal of the second expression.

$\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc} \longleftarrow$ Simplify $\dfrac{ad}{bc}$ if possible ($b \neq 0$, $d \neq 0$, and $c \neq 0$)

Example 8:

Multiply $\dfrac{3x^2y}{2z^2} \cdot \dfrac{8x^3z}{15y^3}$

$\dfrac{3x^2y}{2z^2} \cdot \dfrac{8x^3z}{15y^3}$	
$\dfrac{24x^5yz}{30z^2y^3}$	Multiply the numerator with the numerator and the denominator with the denominator.
$\dfrac{24}{30} \cdot \dfrac{x^5}{1} \cdot \dfrac{y}{y^3} \cdot \dfrac{z}{z^2}$	Separate like terms

$\frac{4}{5}\cdot\frac{x^5}{1}\cdot\frac{1}{y^2}\cdot\frac{1}{z}$	Simplify using integer exponent properties
$\frac{4x^5}{5y^2z}$	Multiply the numerator with the numerator and the denominator with the denominator.

Example 9:

Multiply $\frac{x+1}{x+4}\cdot\frac{x^2+4x}{x+2}$

$\frac{x+1}{x+4}\cdot\frac{x^2+4x}{x+2}$	
$\frac{x+1}{x+4}\cdot\frac{x\left(x+4\right)}{x+2}$	Factor
$\frac{x+1}{\cancel{x+4}}\cdot\frac{x\left(\cancel{x+4}\right)}{x+2}$	Identify the like factors.
$\frac{x+1}{1}\cdot\frac{x}{x+2}$	Simplify the like factors.
$\frac{x\left(x+1\right)}{x+2}$	Multiply the numerator with the numerator and the denominator with the denominator.

Example 10:

Multiply $\frac{x^2-4}{2x+2}\cdot\frac{x^2-2x-3}{x^2+4x+4}$

$\frac{x^2-4}{2x+2}\cdot\frac{x^2-2x-3}{x^2+4x+4}$	
$\frac{\left(x-2\right)\left(x+2\right)}{2\left(x+1\right)}\cdot\frac{\left(x-3\right)\left(x+1\right)}{\left(x+2\right)\left(x+2\right)}$	Factor
$\frac{\left(x-2\right)\cancel{\left(x+2\right)}}{2\cancel{\left(x+1\right)}}\cdot\frac{\left(x-3\right)\cancel{\left(x+1\right)}}{\cancel{\left(x+2\right)}\left(x+2\right)}$	Identify the like factors.
$\frac{\left(x-2\right)}{2}\cdot\frac{\left(x-3\right)}{\left(x+2\right)}$	Simplify the like factors.
$\frac{\left(x-2\right)\left(x-3\right)}{2\left(x+2\right)}$	Multiply the numerator with the numerator and the denominator with the denominator.

Example 11:

Divide $\dfrac{3x^6yz^2}{7xy^3} \div \dfrac{15xy^3z^8}{7x^6y^2z^6}$

$\dfrac{3x^6yz^2}{7xy^3} \div \dfrac{15xy^3z^8}{7x^6y^2z^6}$	
$\dfrac{3x^6yz^2}{7xy^3} \cdot \dfrac{7x^6y^2z^6}{15xy^3z^8}$	Multiply by the reciprocal of the term that follows the division symbol.
$\dfrac{21x^{12}y^3z^8}{105x^2y^6z^8}$	Multiply the numerator with the numerator and the denominator with the denominator.
$\dfrac{21}{105} \cdot \dfrac{x^{12}}{x^2} \cdot \dfrac{y^3}{y^6} \cdot \dfrac{z^8}{z^8}$	Separate like terms
$\dfrac{1}{5} \cdot \dfrac{x^{10}}{1} \cdot \dfrac{1}{y^3} \cdot \dfrac{1}{1}$	Simplify using integer exponent properties
$\dfrac{x^{10}}{5y^3}$	Multiply the numerator with the numerator and the denominator with the denominator.

Example 12:

Divide $\dfrac{12x-20}{x^2-4x-21} \div \dfrac{9x^2-25}{3x^2+14x+15}$

$\dfrac{12x-20}{x^2-4x-21} \div \dfrac{9x^2-25}{3x^2+14x+15}$	
$\dfrac{x^2-4x-21}{12x-20} \cdot \dfrac{3x^2+14x+15}{9x^2-25}$	
$\dfrac{4(3x-5)}{(x-7)(x+3)} \cdot \dfrac{(x+3)(3x+5)}{(3x-5)(3x+5)}$	Factor
$\dfrac{4\cancel{(3x-5)}}{(x-7)\cancel{(x+3)}} \cdot \dfrac{\cancel{(x+3)}\cancel{(3x+5)}}{\cancel{(3x-5)}\cancel{(3x+5)}}$	Identify the like factors.
$\dfrac{4}{x-7} \cdot \dfrac{1}{1}$	Simplify the like factors.
$\dfrac{4}{x-7}$	Multiply the numerator with the numerator and the denominator with the denominator.

Practice Exercises D

Perform the indicated operation, if possible, simplify. Determine if your answer is a rational expression.

1. $\frac{8x^2}{9y}\cdot\frac{3y^2}{2x^5}$

2. $\frac{4x^6}{3y^7}\cdot\frac{9y^2}{2x^3}$

3. $\frac{-4x^3}{y^4}\div\frac{-2}{x^2y^4}$

4. $\frac{-1}{y^4z^3}\div\frac{6x^2y}{z^2}$

5. $\frac{6y^2}{5x^2}\div\frac{3y^2}{4x^6}$

6. $\frac{8-x}{6x-18}\cdot\frac{3x-9}{2x-16}$

7. $\frac{x+5}{x-6}\cdot\frac{2x+12}{x^2-25}$

8. $\frac{x+2}{x-6}\cdot\frac{3x^2}{x^2+4x+4}$

9. $\frac{x^2+5x-14}{3x^3-6x^2}\cdot\frac{2x^2+6x}{x^2+10x+21}$

10. $\frac{x^2-2x}{x^2-1}\cdot\frac{4x-4}{x^2-4}$

11. $\frac{x^2-2x-24}{4x^2+13x-12}\cdot\frac{8x-6}{x^2-6x}$

12. $\frac{2x^2+19x-10}{x^2+x-12}\cdot\frac{x^2-16}{x^2-100}$

13. $\frac{x^2+x-6}{x^2+5x+4}\cdot\frac{3x^2+14x+8}{2x^2+7x+3}$

14. $\frac{4x-24}{x^2-6x+5}\div\frac{-6x+36}{x^2-8x+15}$

15. $\frac{x+4}{x^2-36}\div\frac{4x^2+16x}{x^2-4x-12}$

16. $\frac{x^2-9}{2x-2}\div\frac{x^2-2x-3}{x-1}$

17. $\frac{5x^2+5x}{x-4}\div\frac{x^2-4x-5}{x^3-4x^2}$

18. $\frac{2x^2+9x+9}{x^2-8x+12}\div\frac{4x^2-9}{x^2-6x}$

19. $\frac{x^2+3x+2}{3x-18}\div\frac{x^2-1}{x^2-x+30}$

20. $\frac{3x^2+17x+10}{x^2+5x+6}\div\frac{3x+15}{x^2+2x}$

21. $\frac{x^3-64}{x^3+64}\div\frac{x^2-16}{x^2-4x+16}$

Simplify.

22. $\frac{x-1}{x^2-x-6}\div\frac{x^2+4x-5}{x^2+8x+12}\cdot\frac{2x+10}{x+6}$

23. $\frac{2x^2+3x}{x^2-16}\cdot\frac{25x^2-9}{4x^2+12x+9}\div\frac{25x+15}{2x^2+11x+12}$

24. $\frac{4x^2-9}{8x^3-27}\cdot\frac{4x^2+6x+9}{4x^2-8x+3}\div\frac{4x+6}{3x-9}$

25. $\frac{15x^2+5x-50}{32x^2-18}\div\frac{x^2-5x-14}{4x^2+9x-9}\cdot\frac{6x-42}{3x^2+4x-15}$

YOU DECIDE

Are rational expressions closed under multiplication and division? Justify your conclusion using the method of your choice.

Adding and Subtracting Rational Expressions

VOCABULARY

To add a rational expression to another, find a common denominator then add the numerators.

$$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b}=\frac{ad}{bd}+\frac{cb}{bd}=\frac{ad+cb}{bd} \longleftarrow \text{ Simplify } \frac{ad+cb}{bd} \text{ if possible } (b\neq 0 \text{ and } d\neq 0)$$

To subtract one rational expression from another, find a common denominator then subtract the numerators.

$$\frac{a}{b}-\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{d}-\frac{c}{d}\cdot\frac{b}{b}=\frac{ad}{bd}-\frac{cb}{bd}=\frac{ad-cb}{bd} \longleftarrow \text{ Simplify } \frac{ad-cb}{bd} \text{ if possible } (b\neq 0 \text{ and } d\neq 0)$$

The **least common multiple** (LCM) for expressions is the smallest (non-zero) expression that is a multiple of two or more expressions.

Let us review how to add rational numbers. Add $\frac{1}{6}$ and $\frac{2}{15}$.

$\frac{1}{6}+\frac{2}{15}$	You need to find a common denominator. The least common multiple of 6 and 15 is 30. This will be the common denominator.
$\frac{1}{6}\cdot\frac{5}{5}+\frac{2}{15}\cdot\frac{2}{2}$	$30\div 6=5$ so multiply $\frac{1}{6}$ by $\frac{5}{5}$. $30\div 15=2$ so multiply $\frac{2}{15}$ by $\frac{2}{2}$.
$\frac{5}{30}+\frac{4}{30}=\frac{9}{30}$	Add the numerators.
$\frac{9}{30}=\frac{3}{10}$	Simplify.

Adding and subtracting rational expressions is similar to adding and subtracting rational numbers. The key is finding the common denominator or the least common multiple of the denominators.

Example 13:
Find the least common multiple for the expressions.

a. $10x^2y$ and $12y^3$ b. $x+1$ and x^2-1 c. $x^2+11x+24$ and $x^2+15x+56$

a. $10x^2y$ and $12y^3$	
$10x^2y=2\cdot 5\cdot x\cdot x\cdot y$ $12y^3=2\cdot 2\cdot 3\cdot y\cdot y\cdot y$	Find the prime factorizations of each expression.
$10x^2y=2\cdot 5\cdot x\cdot x\cdot y$ $12y^3=2\cdot 2\cdot 3\cdot y\cdot y\cdot y$	Identify what factors the expressions have in common $(2y)$. Then identify what is unique to both expressions $(5x^2 \text{ and } 6y^2)$.
$(2y)(5x^2)(6y^2)=60x^2y^3$	Multiply the common factors and the unique factors to obtain the least common multiple.

b. $x+1$ and x^2-1	
$x-1=(x-1)$ $x^2-1=(x-1)(x+1)$	Factor each expression. Identify what factors the expressions have in common $(x+1)$. Then identify what is unique to both expressions $(x-1)$.
$(x+1)(x-1)=x^2-1$	Multiply the common factors and the unique factors to obtain the least common multiple.

c. $x^2+11x+24$ and $x^2+15x+56$	
$x^2+11x+24=(x+3)(x+8)$ $x^2+15x+56=(x+7)(x+8)$	Factor each expression. Identify what factors the expressions have in common $(x+8)$. Then identify what is unique to both expressions $(x+3)$ and $(x+7)$.
$(x+8)(x+3)(x+7)=x^3+18x^2+101x+168$	Multiply the common factors and the unique factors to obtain the least common multiple.

Example 14:
Perform the indicated operation. If possible, simplify.

a. $\frac{3}{7x}+\frac{4}{7x}$

b. $\frac{x-3}{x-5}-\frac{7-x}{x-5}$

c. $\frac{x^2}{x+2}-\frac{4}{x+2}$

a. $\frac{3}{7x}+\frac{4}{7x}$	
$\frac{3+4}{7x}=\frac{7}{7x}$	Add the numerators.
$\frac{7}{7x}=\frac{1}{x}$	Simplify.

b. $\frac{x-3}{x-5}-\frac{7-x}{x-5}$	
$\frac{(x-3)-(7-x)}{x-5}=\frac{2x-10}{x-5}$	Subtract the numerators.
$\frac{2x-10}{x-5}=\frac{2(x-5)}{x-5}=2$	Simplify.

c. $\frac{x^2}{x+2}-\frac{4}{x+2}$	
$\frac{x^2-4}{x+2}$	Subtract the numerators.
$\frac{x^2-4}{x+2}=\frac{(x-2)(x+2)}{x+2}=x-2$	Simplify.

Example 15:
Perform the indicated operations. If possible, simplify.

a. $\frac{1}{x+4}+\frac{8}{x^2-16}$ b. $\frac{3}{x+1}-\frac{2}{x-3}$ c. $\frac{x+2}{x^2-5x+4}-\frac{x}{x^2-3x-4}$

a. $\frac{1}{x+4}+\frac{8}{x^2-16}$	
$\frac{1}{x+4}+\frac{8}{(x+4)(x-4)}$	Factor and determine the LCM. The LCM is $(x+4)(x-4)$.
$\frac{1}{x+4}\cdot\frac{x-4}{x-4}+\frac{8}{(x+4)(x-4)}\cdot\frac{1}{1}$ $\frac{x-4}{(x+4)(x-4)}+\frac{8}{(x+4)(x-4)}$	$\frac{(x+4)(x-4)}{x+4}=x-4$ Multiply the first expression by $\frac{x-4}{x-4}$. $\frac{(x+4)(x-4)}{(x+4)(x-4)}=1$ Multiply the second expression by $\frac{1}{1}$.
$\frac{x-4+8}{(x+4)(x-4)}=\frac{x+4}{(x+4)(x-4)}$	Add the numerators.
$\frac{x+4}{(x+4)(x-4)}=\frac{1}{x-4}$	Simplify.
$\frac{1}{x+4}+\frac{8}{x^2-16}=\frac{1}{x-4}$	

b. $\frac{3}{x+1}-\frac{2}{x-3}$	
$\frac{3}{x+1}-\frac{2}{x-3}$	There are no common factors so the LCM is $(x+1)(x-3)$.
$\frac{3}{x+1}\cdot\frac{x-3}{x-3}-\frac{2}{x-3}\cdot\frac{x+1}{x+1}$ $\frac{3(x-3)}{(x+1)(x-3)}-\frac{2(x+1)}{(x-3)(x+1)}$	$\frac{(x+1)(x-3)}{x+1}=x-3$ Multiply the first expression by $\frac{x-3}{x-3}$. $\frac{(x+1)(x-3)}{x-3}=x+1$ Multiply the second expression by $\frac{x+1}{x+1}$.
$\frac{3x-9-(2x+2)}{(x+1)(x-3)}=\frac{3x-9-2x-2}{(x+1)(x-3)}$	Subtract the numerators.
$\frac{x-11}{(x+1)(x-3)}$	Simplify.
$\frac{3}{x+1}-\frac{2}{x-3}=\frac{x-11}{(x+1)(x-3)}$	

c. $\frac{x+2}{x^2-5x+4}-\frac{x}{x^2-3x-4}$	
$\frac{x+2}{(x-1)(x-4)}-\frac{x}{(x+1)(x-4)}$	Factor and determine the LCM. The LCM is $(x-4)(x-1)(x+1)$.
$\frac{x+2}{(x-1)(x-4)}\cdot\frac{x+1}{x+1}-\frac{x}{(x+1)(x-4)}\cdot\frac{x-1}{x-1}$ $\frac{x^2+3x+2}{(x-1)(x-4)(x+1)}-\frac{x^2-x}{(x+1)(x-4)(x-1)}$	$\frac{(x-4)(x-1)(x+1)}{(x-1)(x-4)}=x+1$ Multiply the first expression by $\frac{x+1}{x+1}$. $\frac{(x-4)(x-1)(x+1)}{(x+1)(x-4)}=x-1$ Multiply the second expression by $\frac{x-1}{x-1}$.
$\frac{x^2+3x+2-(x^2-x)}{(x-1)(x-4)(x+1)}$	Subtract the numerators.
$\frac{4x+2}{(x-1)(x-4)(x+1)}$	Simplify.
$\frac{x+2}{x^2-5x+4}-\frac{x}{x^2-3x-4}=\frac{4x+2}{(x-1)(x-4)(x+1)}$	

Practice Exercises E

Perform the indicated operation, if possible, simplify. Determine if your answer is a rational expression.

1. $\frac{27}{11x}-\frac{5}{11x}$

2. $\frac{14x}{2x-3}-\frac{21}{2x-3}$

3. $\frac{2x}{x+1}+\frac{2}{x+1}$

4. $\frac{3x}{x^2+6x+9}+\frac{9}{x^2+6x+9}$

5. $\frac{5x}{25x^2-49}-\frac{7}{25x^2-49}$

6. $\frac{3}{x+2}-\frac{2}{3x+6}$

7. $\frac{6}{x+1}-\frac{x}{2x+2}$

8. $\frac{5}{4x-2}+\frac{1}{10x-5}$

9. $\frac{7}{6x-30}+\frac{2}{3x-15}$

10. $\frac{2}{5x}-\frac{3}{20x^2}$

11. $\frac{3}{x^2-3x+2}-\frac{3}{x-2}$

12. $\frac{x-3}{2x-1}+\frac{x+5}{2x^2+9x-5}$

13. $\frac{1}{x+4}-\frac{3}{x^2+11x+28}$

14. $\frac{2}{3x+1}-\frac{5}{x-4}$

15. $\frac{x}{x-7}+\frac{7}{x+7}$

16. $\frac{2x}{x+5}-\frac{x}{x+8}$

17. $\frac{-6}{x-3}+\frac{5}{x-2}$

18. $\frac{4}{x^2-x}+\frac{6}{x^2-4x+3}$

19. $\frac{3}{x^2-4}+\frac{2}{x^2+5x+6}$

20. $\frac{3}{2x^2-14x}+\frac{2}{x^2-8x+7}$

21. $\frac{4}{x^2-25}-\frac{4}{x^2+10x+25}$

Simplify.

22. $\frac{4}{x+3}-\frac{10}{x-3}+\frac{x+2}{x^2-9}$

23. $\frac{8}{2x+3}+\frac{4}{x-2}-\frac{3x}{2x^2-x-6}$

24. $\frac{5}{2x}+\frac{4}{3y}+\frac{11}{6xy}$

25. $\frac{2}{x}-\frac{3}{x-1}-\frac{1}{x+2}$

YOU DECIDE

Are rational expressions closed under addition and subtraction? Justify your conclusion using the method of your choice.

Unit 2 Cluster 8 (A.REI.2): One Variable Rational and Radical Equations

Cluster 8: Understand solving equations as a process of reasoning and explain the reasoning.

- 2.8 Solve simple rational equations in one variable and give examples of how extraneous solutions may arise.
- 2.8 Solve simple radical equations in one variable and give examples of how extraneous solutions may arise.

VOCABULARY

A **rational equation** is an equation that contains one or more rational expressions (i.e., $\frac{3}{x+2}+\frac{1}{x-2}=\frac{x}{x^2-4}$ is a rational equation).

An **extraneous solution** is a solution of an equation that has been transformed or derived from the original equation but it is not a solution of the original equation. When working with rational functions you must check the solution in the original equation.

Finding Restrictions on Rational Equations

The denominator of a rational expression cannot be zero. When solving a rational equation, any values that would make any denominator zero must be excluded as possible answers. These are referred to as restrictions.

Example 1:
Find the restrictions for each rational equation.

a. $\frac{2x+1}{x+5}=1$

b. $\frac{-2}{x+8}=\frac{4x+3}{2x^2+15x-8}$

a. $\frac{2x+1}{x+5}=1$	
$x+5=0$ $x=-5$	Set the denominator equal to zero and solve for *x*.
The possible answers cannot include $x=-5$.	

b. $\frac{-2}{x+8}=\frac{4x+3}{2x^2+15x-8}$	
$\frac{-2}{x+8}=\frac{4x+2}{(2x-1)(x+8)}$	Factor the denominator of the expression on the right side.

$2x-1=0$ $\quad$ $x+8=0$ $2x=1$ $\quad$ $x=-8$ $x=\frac{1}{2}$	Set each unique factor equal to zero and solve for *x*. Notice that the factor $x+8$ is repeated.
The possible answers cannot include $x=-8$ and $x=\frac{1}{2}$.	

Practice Exercises A

Find the restrictions for each rational equation.

1. $\frac{3x+4}{x+9}=4$

2. $\frac{3}{2x+3}=\frac{1}{x-3}$

3. $\frac{2x}{x+6}=\frac{x}{x-1}$

4. $\frac{x+1}{x^2+3x-40}=\frac{1}{x-5}$

5. $\frac{2x-1}{x^2}=\frac{1}{x}$

6. $\frac{x^2-8x-9}{x^2+2}=1$

7. $\frac{x+5}{2x^2-2x}=\frac{2}{x}$

8. $\frac{x^2-9}{2x^2-21x+54}=\frac{1}{2}$

9. $\frac{x^2-5x-24}{3x^2-28x-20}=\frac{1}{3}$

Solving Rational Equations

To solve a rational equation:

- determine any values that would make the denominator zero
- find a common denominator by finding the least common multiple of the denominators
- multiply all of the terms on both sides of the equation by the common denominator to eliminate the fractions
- simplify and solve the resulting equation
- compare your answer with the restrictions to ensure that it is valid

Example 2:

Solve $\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$.

$\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$	
$x=0$	Determine any numbers that will make the denominator zero. The answer cannot be $x=0$.
$\frac{2}{3}\cdot\frac{6x}{1}-\frac{1}{x}\cdot\frac{6x}{1}=\frac{5}{6}\cdot\frac{6x}{1}$ $\frac{12x}{3}-\frac{6x}{x}=\frac{30x}{6}$	The common denominator is $6x$. Multiply each term by $6x$ and simplify.
$4x-6=5x$ $-6=x$	Simplify and solve the equation.

Example 3:

Solve $\frac{3}{x+2}+\frac{1}{x-2}=\frac{x}{x^2-4}$.

$\frac{3}{x+2}+\frac{1}{x-2}=\frac{x}{x^2-4}$	
$\frac{3}{x+2}+\frac{1}{x-2}=\frac{x}{(x+2)(x-2)}$	Factor the denominators.
$x+2=0$ $\quad$ $x-2=0$ $x=-2$ $\quad$ $x=2$	Find any values that will make the denominators zero. The answer cannot be $x=-2$ or $x=2$.
The common denominator is $(x+2)(x-2)$. Multiply each term by $(x+2)(x-2)$ and simplify. $\frac{3}{\cancel{x+2}}\cdot\frac{\cancel{(x+2)}(x-2)}{1}+\frac{1}{\cancel{x-2}}\cdot\frac{(x+2)\cancel{(x-2)}}{1}=\frac{x}{\cancel{(x-2)}\cancel{(x+2)}}\cdot\frac{\cancel{(x+2)}\cancel{(x-2)}}{1}$	
$3(x-2)+1(x+2)=x$ $3x-6+x+2=x$ $4x-4=x$ $-4=-3x$ $\frac{4}{3}=x$	Simplify and solve the new equation. Compare the answer against the restrictions to make sure that it is valid.

Example 4:

$\frac{2}{x-1}+\frac{2}{x+2}=1$

$\frac{2}{x-1}+\frac{2}{x+2}=1$	
$x-1=0$ $\quad$ $x+2=0$ $x=1$ $\quad$ $x=-2$	Find any values that will make the denominators zero. The answer cannot be $x=1$ or $x=-2$.
The common denominator is $(x-1)(x+2)$. Multiply each term by $(x-1)(x+2)$ and simplify. $\frac{2}{\cancel{x-1}}\cdot\frac{\cancel{(x-1)}(x+2)}{1}+\frac{2}{\cancel{x+2}}\cdot\frac{(x-1)\cancel{(x+2)}}{1}=1\cdot\frac{(x-1)(x+2)}{1}$	
$2(x+2)+2(x-1)=(x-1)(x+2)$ $2x+4+2x-2=x^2+x-2$ $4x+2=x^2+x-2$ $0=x^2-3x-4$ $0=(x-4)(x+1)$ $0=x-4$ $\quad$ $0=x+1$ $4=x$ $\quad$ $-1=x$	Simplify and solve the new equation. Compare the answer against the restrictions to make sure that it is valid.

Example 5:

Solve $\frac{5x}{x+1}=4-\frac{5}{x+1}$.

$\frac{5x}{x+1}=4-\frac{5}{x+1}$	
$x+1=0$ $x=-1$	Find any values that will make the denominators zero. The answer cannot be $x=-1$.
$\frac{5x}{\cancel{x+1}}\cdot\frac{\cancel{x+1}}{1}=4\cdot(x+1)-\frac{5}{\cancel{x+1}}\cdot\frac{\cancel{x+1}}{1}$ $5x=4(x+1)-5$	The common denominator is $(x+1)$. Multiply each term by $(x+1)$ and simplify.
$5x=4x+4-5$ $5x=4x-1$ $x=-1$	Simplify and solve the equation.
The mathematical answer is $x=-1$, but this value will make the denominator zero, therefore, $x=-1$ is an extraneous solution and there is no solution to this equation.	

Practice Exercises B

Solve each equation.

1. $\frac{11}{3x}-\frac{1}{3}=\frac{-4}{x^2}$

2. $\frac{3}{2x}-\frac{5}{3x}=2$

3. $\frac{1}{4x}-\frac{3}{4}=\frac{7}{x}$

4. $\frac{2}{4}-\frac{3}{2x}=\frac{1}{x}$

5. $\frac{x}{x-7}=\frac{49}{x^2-7x}$

6. $\frac{3}{x-1}+\frac{3}{10}=\frac{5}{2x-2}$

7. $\frac{x}{x+1}+\frac{5}{x}=\frac{1}{x^2+x}$

8. $\frac{4}{x^2+4x-5}+\frac{7}{x+5}=\frac{5}{x-1}$

9. $\frac{4}{x^2-9}-\frac{2}{x+3}=\frac{3}{2x-6}$

10. $\frac{x}{x-2}+\frac{3}{x-1}=1$

11. $\frac{2x}{x+3}=1-\frac{6}{x+3}$

12. $\frac{2}{x+3}+\frac{3}{x}=\frac{10}{x^2+3x}$

13. $\frac{8}{x^2+8x+12}=\frac{4}{x+6}+\frac{4}{x+2}$

14. $\frac{4}{x-2}-\frac{2}{x^2-4}=\frac{6}{x+2}$

15. $\frac{5}{x^2-7x+12}-\frac{2}{3-x}=\frac{5}{x-4}$

16. $\frac{6}{x+1}=\frac{x}{x-1}$

17. $\frac{4}{x-7}=\frac{-2x}{x+3}$

18. $\frac{1}{x+6}=\frac{36}{x^2+6x}$

19. $\frac{5x}{x-2}=7+\frac{10}{x-2}$

20. $\frac{3}{x-2}=\frac{5}{x+4}$

21. $x-\frac{12}{x}=4$

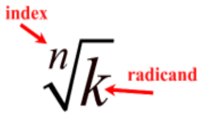
22. $\frac{2}{x-4}+\frac{1}{x}=\frac{x}{4-x}$

23. $\frac{5}{4x}=\frac{7}{5x-2}$

24. $\frac{x+3}{x+2}-\frac{x}{x^2-4}=\frac{x}{x-2}$

VOCABULARY

A **radical equation** is an equation that has a variable in a radicand or a variable with a rational exponent (i.e, $\sqrt{2x+3}=4$ or $(4x-1)^{1/3}=1$). The **radicand** is the expression under the radical sign. The **index** is the small number outside of the radical sign.



Solving Radical Equations

To solve a radical equation:

- isolate the radical on one side of the equation
- raise each side to the power of the index
- simplify
- check solutions in the original equation to eliminate any extraneous solutions

Example 6:

Solve $4+\sqrt{3x+10}=9$.

$4+\sqrt{3x+10}=9$	
$\sqrt{3x+10}=5$	Isolate the radical term by subtracting 4 from each side.
$\left(\sqrt{3x+10}\right)^2=5^2$ $3x+10=25$	Square each side of the equation.
$3x=15$ $x=5$	Solve for *x*.
$4+\sqrt{3\cdot 5+10}\overset{?}{=}9$ $4+\sqrt{15+10}\overset{?}{=}9$ $4+\sqrt{25}\overset{?}{=}9$ $4+5\overset{?}{=}9$ $9=9$	Check the solution in the original equation.

Example 7:

Solve $-1+\sqrt[3]{2x-5}=2$.

$-1+\sqrt[3]{2x-5}=2$	
$\sqrt[3]{2x-5}=3$	Isolate the radical term by adding 1 to each side.
$\left(\sqrt[3]{2x-5}\right)^3=3$ $2x-5=27$	Cube each side of the equation.
$2x=32$ $x=16$	Solve for *x*.
$-1+\sqrt[3]{2\cdot 16-5}\overset{?}{=}2$ $-1+\sqrt[3]{32-5}\overset{?}{=}2$ $-1+\sqrt[3]{27}\overset{?}{=}2$ $-1+3\overset{?}{=}2$ $2=2$	Check the solution in the original equation.

Example 8:

Solve $\sqrt{x+9}-7=x$.

$\sqrt{x+9}-7=x$	
$\sqrt{x+9}=x+7$	Isolate the radical term by adding 7 to each side.
$\left(\sqrt{x+9}\right)^2=\left(x+7\right)^2$ $x+9=x^2+14x+49$	Square each side of the equation. Remember that $\left(x+7\right)^2=\left(x+7\right)\left(x+7\right)$.
$0=x^2+13x+40$ $0=\left(x+5\right)\left(x+8\right)$ $x+5=0 \qquad x+8=0$ $x=-5 \qquad x=-8$	Solve for *x*.

$\sqrt{-5+9}-7\overset{?}{=}-5$ $\sqrt{4}-7\overset{?}{=}-5$ $2-7\overset{?}{=}-5$ $-5=-5$	$\sqrt{-8+9}-7\overset{?}{=}-8$ $\sqrt{1}-7\overset{?}{=}-8$ $1-7\overset{?}{=}-8$ $-6\neq-8$	Check the solutions in the original equation.
The only solution is $x=-5$ because $x=-8$ does not work in the original equation so it is an extraneous solution.		

Example 9:

Solve $\sqrt{3x+1}-\sqrt{x+1}=2$.

$\sqrt{3x+1}-\sqrt{x+1}=2$		
$\sqrt{3x+1}=2+\sqrt{x+1}$		Isolate one of the radical terms.
$\left(\sqrt{3x+1}\right)^2=\left(2+\sqrt{x+1}\right)^2$ $3x+1=4+4\sqrt{x+1}+x+1$ $3x+1=4\sqrt{x+1}+x+5$		Square each side of the equation. Remember that $\left(2+\sqrt{x+1}\right)^2=\left(2+\sqrt{x+1}\right)\left(2+\sqrt{x+1}\right)$.
$2x-4=4\sqrt{x+1}$		Isolate the radical term.
$(2x-4)^2=\left(4\sqrt{x+1}\right)^2$ $4x^2-16x+16=16(x+1)$		Square each side of the equation. Remember that $(2x-4)^2=(2x-4)(2x-4)$.
$4x^2-16x+16=16x+16$ $4x^2-32x=0$ $4x(x-8)=0$ $4x=0$ $\quad$ $x-8=0$ $x=0$ $\quad$ $x=8$		Solve for *x*.
$\sqrt{3\cdot0+1}-\sqrt{0+1}\overset{?}{=}2$ $\sqrt{1}-\sqrt{1}\overset{?}{=}2$ $0\neq2$	$\sqrt{3\cdot8+1}-\sqrt{8+1}\overset{?}{=}2$ $\sqrt{25}-\sqrt{9}\overset{?}{=}2$ $5-3\overset{?}{=}2$ $2=2$	Check the solutions in the original equation.
The only solution is $x=8$ because $x=0$ does not work in the original equation so it is an extraneous solution.		

Practice Exercises C

Solve each radical equation.

1. $\sqrt{x-2}+5=8$
2. $\sqrt[3]{x-2}+1=4$
3. $2\sqrt{x+4}-5=-3$
4. $\sqrt[4]{x-10}+5=8$
5. $\sqrt{2x-1}-3=2$
6. $\sqrt[5]{x-1}+4=5$
7. $3\sqrt{x}-4=11$
8. $\sqrt[5]{x+3}+7=5$
9. $\sqrt{3x+4}+6=13$
10. $\sqrt[4]{x+5}-7=-5$
11. $\sqrt{x+3}-2=4$
12. $-5\sqrt[3]{x}-9=11$
13. $-\sqrt{x-4}+3=-1$
14. $\sqrt[4]{2x+3}-2=1$
15. $\sqrt{x-2}+4=2$
16. $\sqrt{3x+7}+1=x$
17. $\sqrt[3]{3x+4}+1=2$
18. $\sqrt[3]{6x+9}+8=5$
19. $2\sqrt[3]{x}+6=-4$
20. $\sqrt{11x+3}=2x$
21. $3\sqrt[5]{x+6}-7=-4$
22. $\sqrt[3]{21x+55}-2=8$
23. $\sqrt{x+7}=x-5$
24. $\sqrt[3]{x+3}-8=-6$
25. $\sqrt{4x-3}=2+\sqrt{2x-5}$
26. $\sqrt{2x+6}=2+\sqrt{x-1}$
27. $\sqrt{3-x}+\sqrt{x+2}=3$

HONORS

Recall that when a rational number is multiplied by its reciprocal the product is 1 (i.e., $\frac{a}{b}\cdot\frac{b}{a}=\frac{ab}{ab}=1$). To solve radical equations of the form $x^{\frac{a}{b}}=k$, raise each side of the equation to the power of the reciprocal $\frac{b}{a}$.

Example 10:

Solve $2\left(x+1\right)^{2/3}+5=13$.

$2\left(x+1\right)^{2/3}+5=13$	
$2\left(x+1\right)^{2/3}=8$ $\left(x+1\right)^{2/3}=4$	Isolate the radical term.

$\left((x+1)^{2/3}\right)^{3/2}=4^{3/2}$ $x+1=\left(\sqrt{4}\right)^3$ $x+1=\left(\pm2\right)^3$ $x+1=2^3$ $x+1=8$ $x=7$	$x+1=\left(-2\right)^3$ $x+1=-8$ $x=-9$	The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Raise each side of the equation to the $\frac{3}{2}$ power. Rewrite the using the properties of exponents. Remember that the square root of a number has a positive and a negative solution.
$2(7+1)^{2/3}+5\overset{?}{=}13$ $2(8)^{2/3}+5\overset{?}{=}13$ $2(4)+5\overset{?}{=}13$ $8+5\overset{?}{=}13$ $13=13$	$2(-9+1)^{2/3}+5\overset{?}{=}13$ $2(-8)^{2/3}+5\overset{?}{=}13$ $2(4)+5\overset{?}{=}13$ $8+5\overset{?}{=}13$ $13=13$	Check the solutions in the original equation.
Both $x=7$ and $x=-9$ are solutions to the radical equation.		

Example 11:

Solve $4\sqrt[5]{(x+12)^3}-6=26$.

$4\sqrt[5]{(x+12)^3}-6=26$	
$4\sqrt[5]{(x+12)^3}=32$ $\sqrt[5]{(x+12)^3}=8$	Isolate the radical term.
$(x+12)^{3/5}=8$	Rewrite the expression in rational exponent form.
$\left((x+12)^{3/5}\right)^{5/3}=8^{5/3}$ $x+12=\left(\sqrt[3]{8}\right)^5$ $x+12=(2)^5$ $x+12=32$ $x=20$	The reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$. Raise each side of the equation to the $\frac{5}{3}$ power. Rewrite the using the properties of exponents. Remember that the cube root has only one real answer.

$4\sqrt[5]{(20+12)^3}-6\overset{?}{=}26$ $4\sqrt[5]{(32)^3}-6\overset{?}{=}26$ $4\sqrt[5]{32,768}-6\overset{?}{=}26$ $4\cdot 8-6\overset{?}{=}26$ $32-6\overset{?}{=}26$ $26=26$	Check the solution in the original equation.
$x=20$ is a solution to the radical equation.	

Practice Exercises D

Solve each radical equation.

1. $(x+7)^{2/3}+6=10$
2. $2\sqrt[3]{(x+15)^2}-5=13$
3. $5(x-4)^{3/4}-25=15$
4. $(x+9)^{3/4}-15=12$
5. $(x+8)^{3/2}-6=21$
6. $-2\sqrt{(3-x)^3}+5=-11$
7. $\sqrt[3]{(x+2)^5}-10=22$
8. $\frac{1}{3}(23x+13)^{3/5}+2=11$
9. $\frac{1}{2}(x+20)^{4/3}-5=3$

Unit 4 Clusters 3 and 5 (F.IF.7b,e and F.BF.3): Transformations

Cluster 3: Analyze functions using different representations

4.3 Graph basic functions i.e., square root, cube root, piecewise-defined functions, step functions, absolute value functions, exponential, logarithmic, trigonometric functions, and polynomial functions (including linear and quadratic functions) with and without technology.

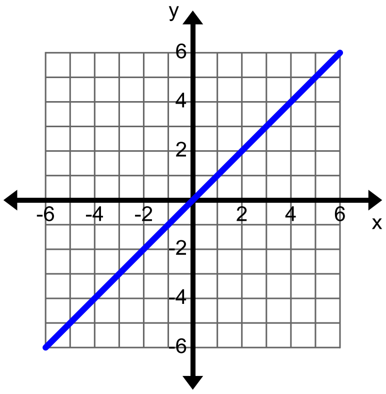
Cluster 5: Build new functions from existing functions

4.5 Identify the effect on the graph by replacing $f\left(x\right)$ with $f\left(x\right)+k$, $kf\left(x\right)$, $f\left(kx\right)$, and $f\left(x+k\right)$ for specific values of *k* (both positive and negative); find the value of *k* given the graphs, with and without technology. Include recognizing even and odd functions graphically and algebraically.

VOCABULARY

There are several types of functions (linear, exponential, quadratic, absolute value, etc.). Each of these could be considered a family with unique characteristics that are shared among the members. The **parent function** is the basic function that is used to create more complicated functions.

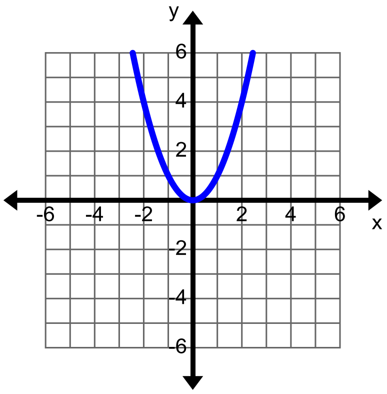
Linear Function

Parent Function $f\left(x\right)=x$	Key Features
	**Domain:** $\left(-\infty,\infty\right)$ **Range:** $\left(-\infty,\infty\right)$ **Intercepts**: *x*-intercept $\left(0,0\right)$, *y*-intercept $\left(0,0\right)$ **Intervals of Increasing/Decreasing:** increasing $\left(-\infty,\infty\right)$ **Intervals where Positive/Negative:** positive $\left(0,\infty\right)$, negative $\left(-\infty,0\right)$ **Relative maximums/minimums**: none **Symmetries:** odd **End Behavior**: right end behavior $\lim\limits_{x\to\infty} x=\infty$; left end behavior $\lim\limits_{x\to-\infty} x=-\infty$

Quadratic Function

Parent Function

$f\left(x\right)=x^2$



Key Features

Domain: $\left(-\infty,\infty\right)$

Range: $\left[0,\infty\right)$

Intercepts: *x*-intercept $\left(0,0\right)$, *y*-intercept $\left(0,0\right)$

Intervals of Increasing/Decreasing: increasing $\left(0,\infty\right)$, decreasing $\left(-\infty,0\right)$

Intervals where Positive/Negative: positive $\left(-\infty,0\right)\cup\left(0,\infty\right)$

Relative maximums/minimums: minimum at $\left(0,0\right)$

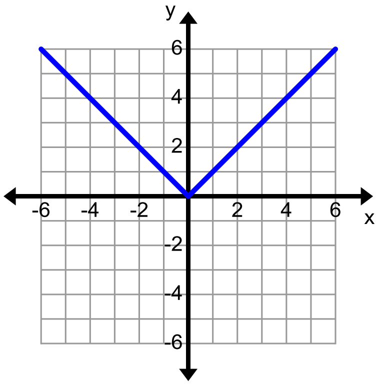
Symmetries: even

End Behavior: right end behavior $\lim\limits_{x\to\infty} x^2=\infty$; left end behavior $\lim\limits_{x\to-\infty} x^2=\infty$

Absolute Value Function

Parent Function

$f\left(x\right)=\left|x\right|$



Key Features

Domain: $\left(-\infty,\infty\right)$

Range: $\left[0,\infty\right)$

Intercepts: *x*-intercept $\left(0,0\right)$, *y*-intercept $\left(0,0\right)$

Intervals of Increasing/Decreasing: increasing $\left(0,\infty\right)$, decreasing $\left(-\infty,0\right)$

Intervals where Positive/Negative: positive $\left(-\infty,0\right)\cup\left(0,\infty\right)$

Relative maximums/minimums: minimum at $\left(0,0\right)$

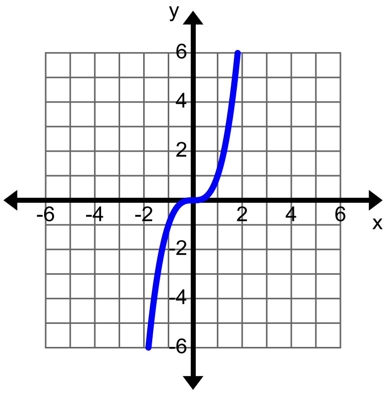
Symmetries: even

End Behavior: right end behavior $\lim\limits_{x\to\infty}\left|x\right|=\infty$; left end behavior $\lim\limits_{x\to-\infty}\left|x\right|=\infty$

Cubic Function

Parent Function

$f(x) = x^3$



Key Features

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Intercepts: x-intercept $(0,0)$, y-intercept $(0,0)$

Intervals of Increasing/Decreasing: increasing $(-\infty, \infty)$

Intervals where Positive/Negative: positive $(0, \infty)$, negative $(-\infty, 0)$

Relative maximums/minimums: none

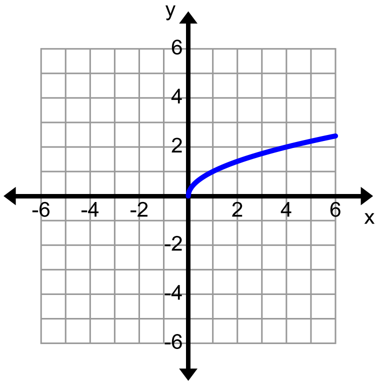
Symmetries: odd

End Behavior: right end behavior $\lim_{x \to \infty} x^3 = \infty$; left end behavior $\lim_{x \to -\infty} x^3 = -\infty$

Square Root Function

Parent Function

$f(x) = \sqrt{x} = x^{1/2}$



Key Features

Domain: $[0, \infty)$

Range: $[0, \infty)$

Intercepts: x-intercept $(0,0)$, y-intercept $(0,0)$

Intervals of Increasing/Decreasing: increasing $(0, \infty)$

Intervals where Positive/Negative: $(0, \infty)$

Relative maximums/minimums: minimum at $(0,0)$

Symmetries: none

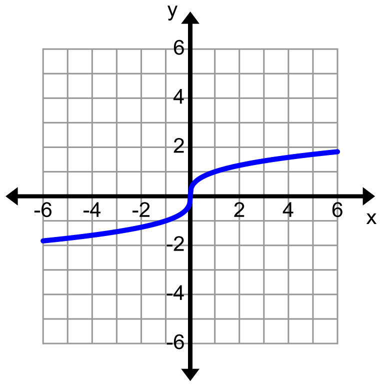
End Behavior: right end behavior $\lim_{x \to \infty} \sqrt{x} = \infty$; left end behavior $\lim_{x \to 0^+} \sqrt{x} = 0$

Cube Root Function

<table>
<tr><td>

Parent Function

$f(x)=\sqrt[3]{x}=x^{1/3}$



</td><td>

Key Features

Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

Intercepts: x-intercept $(0,0)$, y-intercept $(0,0)$

Intervals of Increasing/Decreasing: increasing $(-\infty,\infty)$

Intervals where Positive/Negative: positive $(0,\infty)$, negative $(-\infty,0)$

Relative maximums/minimums: none

Symmetries: odd

End Behavior: right end behavior $\lim_{x\to\infty}\sqrt[3]{x}=\infty$; left end behavior $\lim_{x\to-\infty}\sqrt[3]{x}=-\infty$

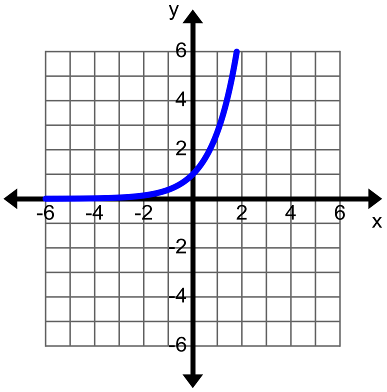
</td></tr>
</table>

Exponential Function

<table>
<tr><td>

Parent Function

$f(x)=a^x$ or $f(x)=e^x$



</td><td>

Key Features

Domain: $(-\infty,\infty)$

Range: $(0,\infty)$

Intercepts: y-intercept $(0,1)$

Intervals of Increasing/Decreasing: increasing $(-\infty,\infty)$

Intervals where Positive/Negative: positive $(-\infty,\infty)$

Relative maximums/minimums: none

Symmetries: none

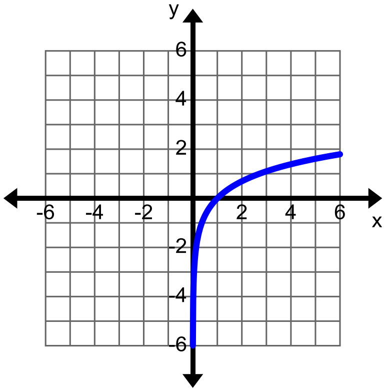
End Behavior: right end behavior $\lim_{x\to\infty}e^x=\infty$; left end behavior $\lim_{x\to-\infty}e^x=0$

</td></tr>
</table>

Logarithmic Function

Parent Function

$f(x)=\log_b x$ or $f(x)=\ln x$



Key Features

Domain: $(0,\infty)$

Range: $(-\infty,\infty)$

Intercepts: x-intercept $(1,0)$

Intervals of Increasing/Decreasing: increasing $(-\infty,\infty)$

Intervals where Positive/Negative: positive $(1,\infty)$, negative $(0,1)$

Relative maximums/minimums: none

Symmetries: none

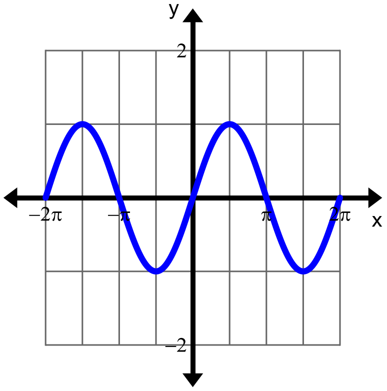
End Behavior: right end behavior $\lim_{x\to\infty}\ln x=\infty$; left end behavior $\lim_{x\to 0^+}\ln x=-\infty$

Note: There is a vertical asymptote at $x=0$.

Sine Function

Parent Function

$f(x)=\sin x$



Key Features

Domain: $(-\infty,\infty)$

Range: $[-1,1]$

Intercepts: x-intercepts $(\pm k\pi,0)$, y-intercept $(0,0)$

Intervals of Increasing/Decreasing: alternating increasing and decreasing in periodic waves

Intervals where Positive/Negative: alternating positive and negative in periodic waves

Relative maximums/minimums: absolute maximum of 1 and absolute minimum of -1

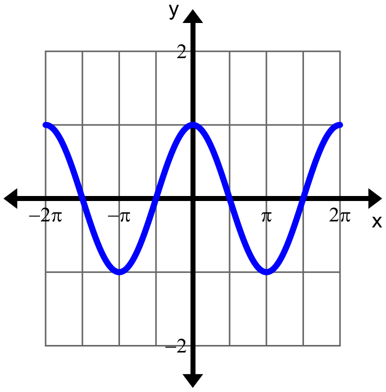
Symmetries: odd

End Behavior: no end behavior because the values oscillate between -1 and 1 and approach no limit

Cosine Function

Parent Function

$f(x) = \cos x$



Key Features

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Intercepts: *x*-intercepts $\left(\pm\frac{k\pi}{2}, 0\right)$ where *k* is odd, *y*-intercept $(0,1)$

Intervals of Increasing/Decreasing: alternating increasing and decreasing in periodic waves

Intervals where Positive/Negative: alternating positive and negative in periodic waves

Relative maximums/minimums: absolute maximum of 1 and absolute minimum of -1

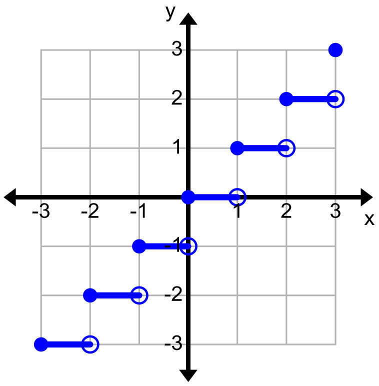
Symmetries: even

End Behavior: no end behavior because the values oscillate between -1 and 1 and approach no limit

Step Functions are piecewise-defined functions made up of constant functions. It is called a step function because the graph resembles a staircase.

Step Function

$f(x) = \text{int } x$



Key Features

Domain: $(-\infty, \infty)$

Range: $\{y \mid y \text{ is an integer}\}$

Intercepts: *x*-intercept $x = [0,1)$ and $y = 0$, *y*-intercept $(0,0)$

Intervals of Increasing/Decreasing: neither increasing nor decreasing

Intervals where Positive/Negative: positive $(1, \infty)$, negative $(-\infty, 0)$

Relative maximums/minimums: none

Symmetries: none

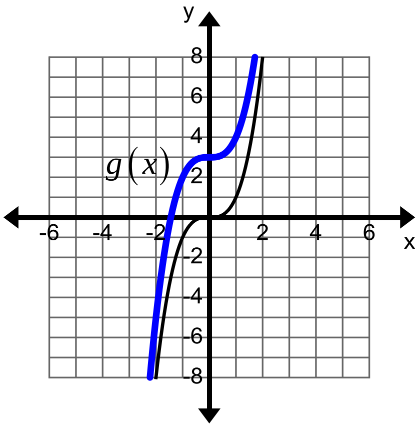
End Behavior: right end behavior $\lim_{x\to\infty} \text{int } x = \infty$; left end behavior $\lim_{x\to-\infty} \text{int } x = -\infty$

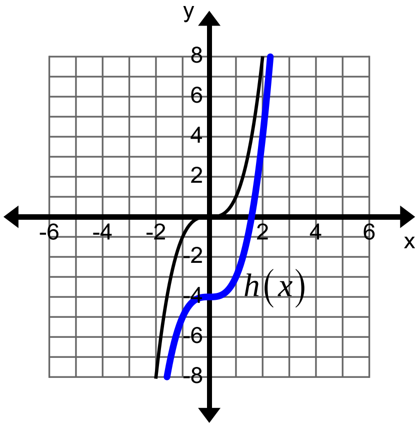
Example 1: Vertical Shift

Given $f(x)=x^3$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=f(x)+3$

b. $h(x)=f(x)-2$

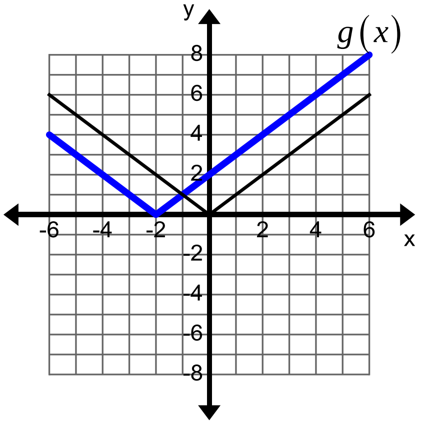
a. $g(x)=f(x)+3$ 	$k=3$ so the graph is shifted up 3 units. The transformed function is neither odd nor even.

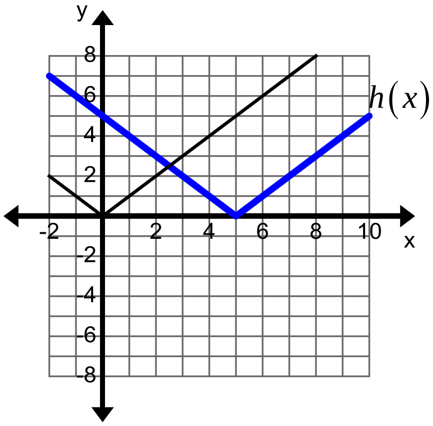
b. $h(x)=f(x)-4$ 	$k=-4$ so the graph is shifted down 4 units. The transformed function is neither odd nor even.

Example 2: Horizontal Shift

Given $f(x)=|x|$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=f(x+2)$ b. $h(x)=f(x-5)$

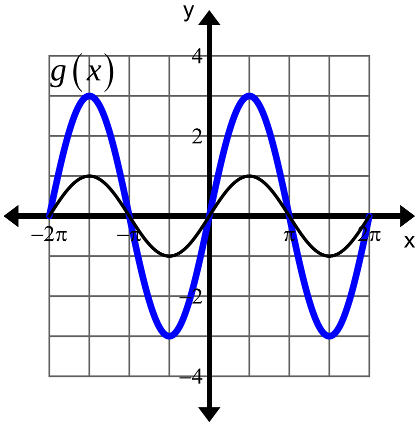
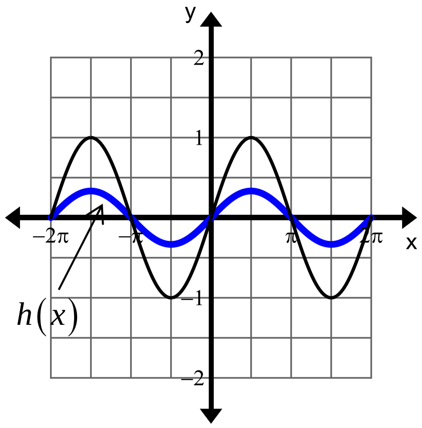
a. $g(x)=f(x+2)$  	$k=2$ so the graph is shifted left 2 units. The transformed function is neither odd nor even.

b. $h(x)=f(x-5)$  	$k=-5$ so the graph is shifted right 5 units. The transformed function is neither odd nor even.

Example 3: Vertical Stretch

Given $f(x)=\cos x$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=3f(x)$ b. $h(x)=\frac{1}{3}f(x)$

a. $g(x)=3f(x)$  	$k=3$ so the graph is stretched vertically by a factor of 3. The transformed function is odd.
b. $h(x)=\frac{1}{3}f(x)$  	$k=\frac{1}{3}$ so the graph is stretched vertically by a factor of $\frac{1}{3}$. The transformed function is odd.

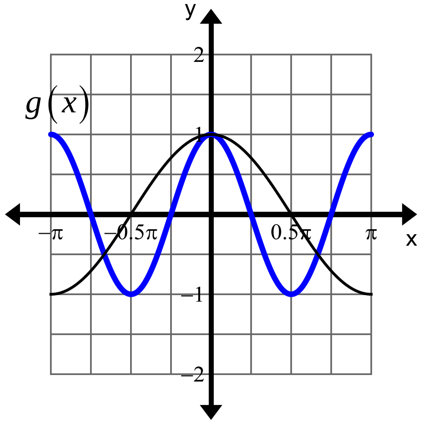
Example 4: Horizontal Stretch

Given $f(x)=\cos x$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=f(2x)$

b. $h(x)=f\left(\frac{1}{2}x\right)$

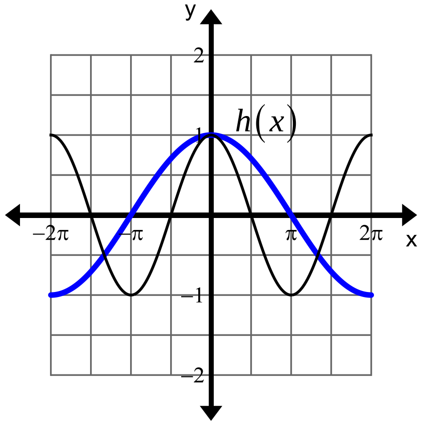
a. $g(x)=f(2x)$



$k=2$ so the graph is stretched horizontally by a factor of $\frac{1}{2}$. Notice that the transformed function has more cycles than the original function.

The transformed function is even.

b. $h(x)=f\left(\frac{1}{2}x\right)$



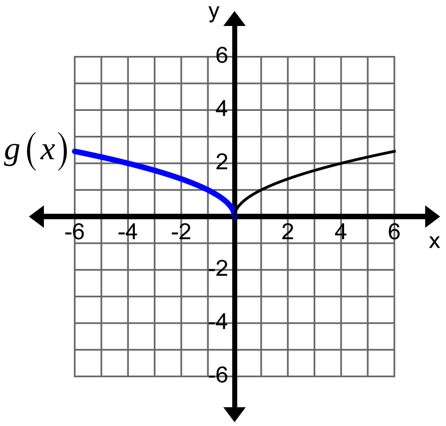
$k=\frac{1}{2}$ so the graph is stretched horizontally by a factor of 2. Notice that the transformed function has fewer cycles than the original function.

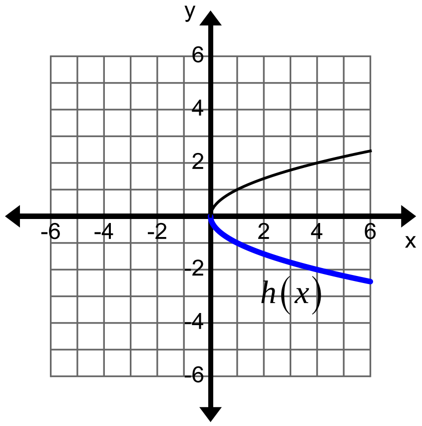
The transformed function is even.

Example 5: Reflections

Given $f(x)=\sqrt{x}$, graph each new function, without technology, and describe the effect of *k* on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=f(-x)$ b. $h(x)=-f(x)$

a. $g(x)=f(-x)$ 	$k=-1$ so the graph is reflected across the *y*-axis. The transformed function is neither odd nor even.

b. $h(x)=-f(x)$ 	$k=-1$ so the graph is reflected across the *x*-axis. The transformed function is neither odd nor even.

Practice Exercises A

Given $f(x)$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

1. $f(x)=x$
   a. $g(x)=-f(x)$
   b. $h(x)=\frac{1}{2}f(x)$
   c. $j(x)=f(x)-5$

2. $f(x)=x^2$
   a. $g(x)=f(x+6)$
   b. $h(x)=f(x)+4$
   c. $j(x)=2f(x)$

3. $f(x)=|x|$
   a. $g(x)=-3f(x)+4$
   b. $h(x)=f(x-7)+2$
   c. $j(x)=\frac{1}{4}f(x+1)$

4. $f(x)=\sqrt{x}$
   a. $g(x)=f(-x)$
   b. $h(x)=-f(x+3)$
   c. $j(x)=2f(x)-3$

5. $f(x)=x^3$
   a. $g(x)=-\frac{1}{4}f(x-2)$
   b. $h(x)=f(x)+6$
   c. $j(x)=f(x+4)$

6. $f(x)=\sqrt[3]{x}$
   a. $g(x)=f(x)-1$
   b. $h(x)=f(x+2)-4$
   c. $j(x)=3f(x-2)$

7. $f(x)=e^x$
   a. $g(x)=f(-x)$
   b. $h(x)=f(x+5)$
   c. $j(x)=f(x)-2$

8. $f(x)=\ln(x)$
   a. $g(x)=2f(-x)$
   b. $h(x)=-f(x)+3$
   c. $j(x)=f(x-1)-1$

9. $f(x)=\sin x$
   a. $g(x)=f(2x)$
   b. $h(x)=-\frac{3}{2}\sin x$
   c. $j(x)=f(-x)$

10. $f(x)=\cos x$
    a. $g(x)=-2f(x)$
    b. $h(x)=f\left(\frac{1}{3}x\right)$
    c. $j(x)=f(x)+2$

11. $f(x)=\text{int}\ \ x$
    a. $g(x)=f(-x)$
    b. $h(x)=f\left(\frac{1}{2}x\right)$
    c. $j(x)=2f(x)$

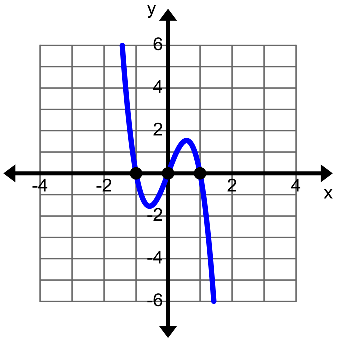
Example 6:

Determine whether each function can be obtained from the parent function, $f(x) = x^n$, using basic transformations. If so, describe the sequence of transformations.

a. $g(x) = -4x^3 + 4x$

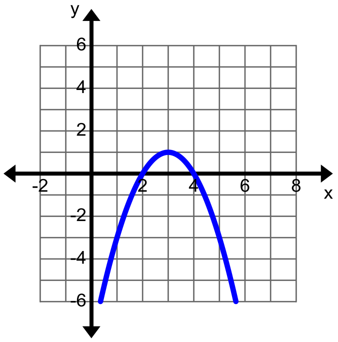
b. $h(x) = x^2 - 6x + 8$

a. $g(x) = -4x^3 + 4x$



By looking that the graph of $g(x)$ you can see that it has three real roots, but the parent function $f(x) = x^3$ has only one real root. Since the parent function is increasing on its entire domain it is not possible to obtain $g(x)$ through basic transformations such as stretches, reflections, and translations.

b. $h(x) = -x^2 + 6x - 8$



By looking at the graph of $h(x)$ you can see that the function has been reflected over the *x*-axis, translated up 1 unit and right 3 units from the parent function $f(x) = x^2$.

You could also have used the process of completing the square rewrite $h(x)$ in vertex form.

$$h(x) = -\left(x^2 - 6x + \_\_\_\right) - 8$$

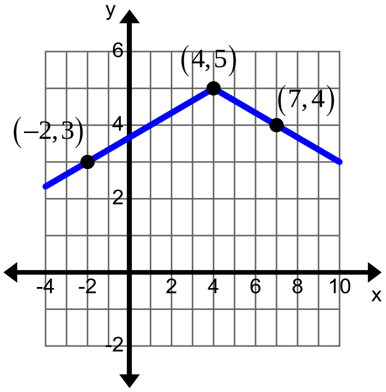
$$h(x) = -\left(x^2 - 6x + \left(-\frac{6}{2}\right)^2\right) - 8 + 9$$

$$h(x) = -(x-3)^2 + 1$$

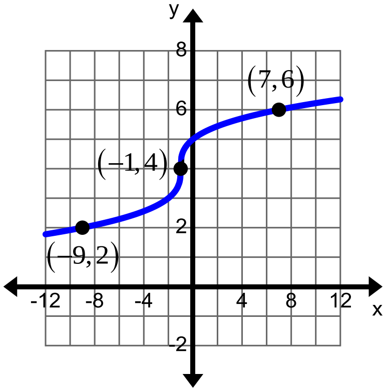
Example 7:

Determine the transformations that were used to change the given parent function to the function that is graphed.

a. $f\left(x\right)=\left|x\right|$



b. $y=\sqrt[3]{x}$



a. The transformed absolute value function opens down so it has been reflected across the *x*-axis. The first point away from the vertex is down 1 and over three so it has been vertically stretched by a factor of $\frac{1}{3}$. The vertex is at $\left(4,5\right)$ so it has been translated up 5 units and right 4 units. • Reflection over the *x*-axis • Vertical stretch by a factor of $\frac{1}{3}$ • Translation of 5 units up and 4 units right	b. The middle of the s-curve is normally at $\left(0,0\right)$ but is now at $\left(-1,4\right)$. The cube root of 8 is 2 so there should be a point at $\left(8,2\right)$, but there is a point at $\left(7,6\right)$ instead. The transformed graph has been translated up 4 units and left 1 unit. • Translation up 4 units and left 1 unit.

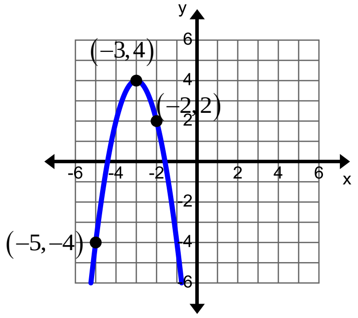
Practice Exercises B

Determine whether each function can be obtained from the parent function, $f(x) = x^n$, using basic transformations. If so, describe the sequence of transformations.

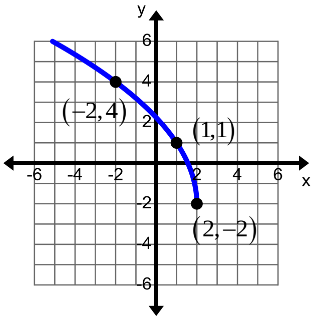
1. $g(x) = x^2 - 4$
2. $g(x) = x^2 - 2x - 15$
3. $g(x) = 4x^2 + 4x - 3$
4. $g(x) = x^3 - 6x^2 + 12x - 8$
5. $g(x) = -2x^3$
6. $g(x) = -3x^3 + 12x$
7. $g(x) = (x+3)^4 - 5$
8. $g(x) = x^4 - 8x^2 + 7$
9. $g(x) = 2x^4 - x^2$
10. $g(x) = -3x^5 + 2$
11. $g(x) = (x+1)^5$
12. $g(x) = (x+5)^5 + 7$

Determine the transformations that were used to change the given parent function to the function that is graphed.

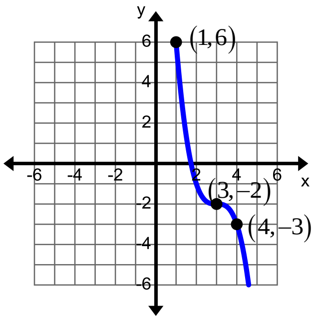
13. $f(x) = x^2$



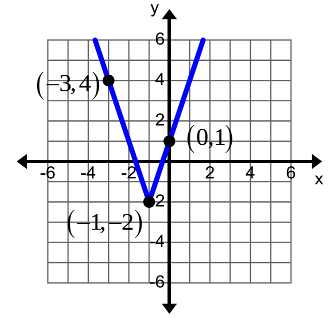
14. $f(x) = \sqrt{x}$



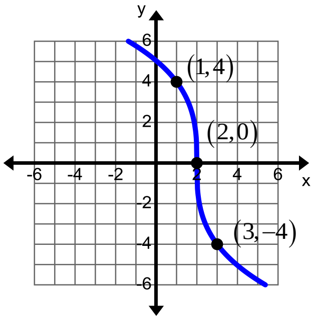
15. $f(x) = x^3$



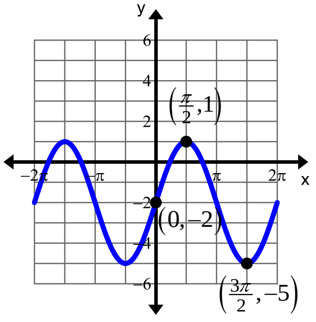
16. $f(x) = |x|$



17. $f(x) = \sqrt[3]{x}$



18. $f(x) = \sin(x)$



Unit 4 Clusters 3 HONORS (F.IF.7d): Graphing Rational Functions

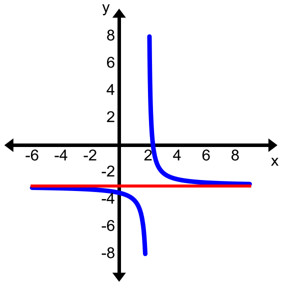
Cluster 3: Analyze functions using different representations

4.3 Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior

VOCABULARY

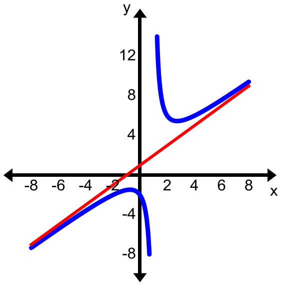
An **asymptote** is a line that a graph approaches, as *x* increases or decreases, but does not intersect. **Horizontal** and **oblique** asymptotes occur with rational functions and model end behavior. Vertical asymptotes occur when the denominator is equal to zero, but the numerator is not zero. The graph may cross a horizontal or an oblique asymptote near the origin.

Horizontal Asymptote



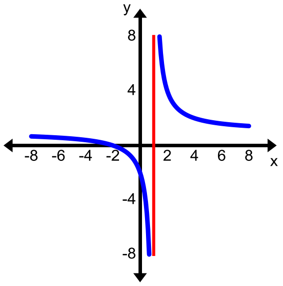
$y = -3$

Oblique Asymptote



$y = x + 1$

Vertical Asymptote

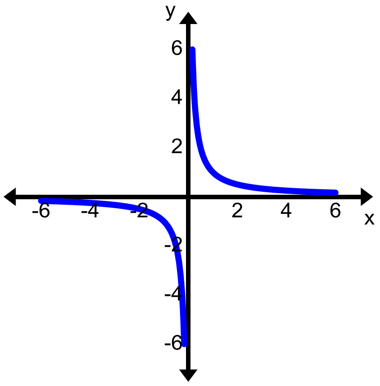


$x = 1$

Reciprocal Function

Parent Function

$$f(x) = \frac{1}{x}$$



Key Features

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

Intercepts: none

Intervals of Increasing/Decreasing: decreasing $(-\infty, 0) \cup (0, \infty)$

Intervals where Positive/Negative: positive $(0, \infty)$, negative $(-\infty, 0)$

Relative maximums/minimums: none

Symmetries: odd

End Behavior: right end behavior $\lim_{x \to \infty} x = 0$; left end behavior $\lim_{x \to -\infty} x = 0$

Let $p(x)$ and $q(x)$ be polynomials with no common factors other than 1. The graph of the rational function $f(x)=\frac{p(x)}{q(x)}=\frac{a_m x^m + a_{m-1}x^{m-1}+...+a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1}+...+b_1 x + b_0}$ has the following characteristics:

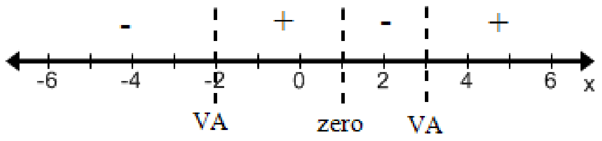
1. The *x*-intercepts of the graph of *f* are the real zeros of numerator.
2. The graph of *f* has a vertical asymptote at each real zero of the denominator unless the numerator and the denominator share a factor.
3. The graph of *f* has at most one horizontal asymptote.
    - If the degree of $p(x)$ is less than the degree of $q(x)$, then the line $y=0$ is the horizontal asymptote.
    - If the degree of $p(x)$ is equal to the degree of $q(x)$, then the line $y=\frac{a}{b}$ is the horizontal asymptote.
    - If degree of $p(x)$ is greater than the degree of $q(x)$, then there may be an oblique asymptote which can be found using long division. The oblique asymptote will be in the form $y=mx+b$.

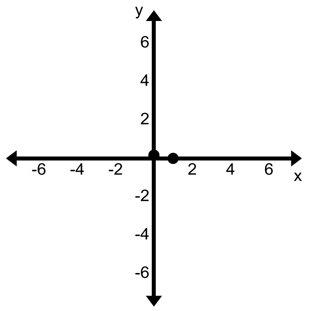
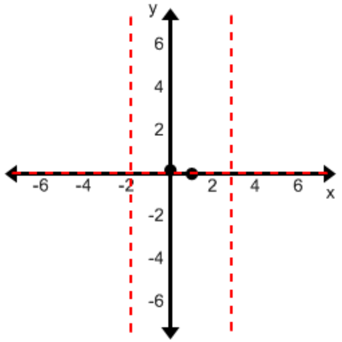
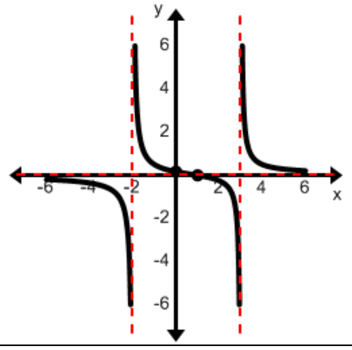
Sketching Rational Functions

1. Factor the numerator and the denominator.
2. Find the *x*-intercept and *y*-intercepts.
3. Find the horizontal or oblique and vertical asymptotes.
4. Create a sign array to determine where the function is positive or negative.
5. Sketch the graph.
6. Identify the domain and range.

Example 1:

Graph the function $f(x)=\dfrac{x-1}{x^2-x-6}$ and identify the domain and range.

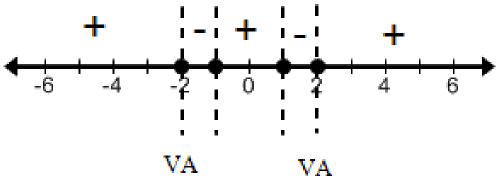
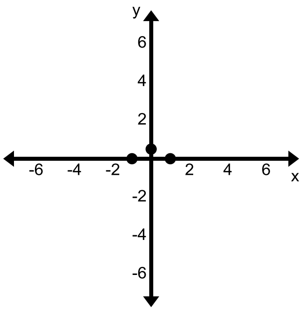
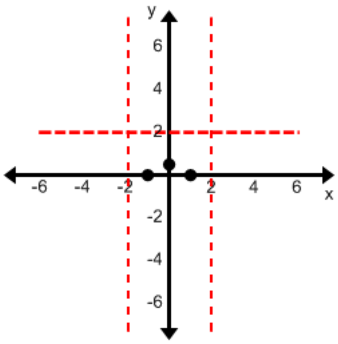
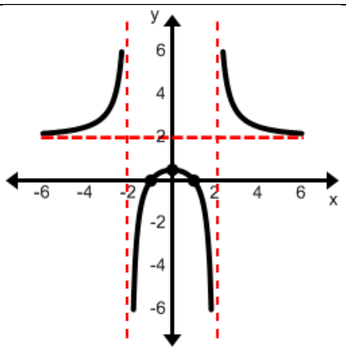
<table>
<tr><td colspan="2">$f(x)=\dfrac{x-1}{(x-3)(x+2)}$</td><td>Factor the numerator and the denominator.</td></tr>
<tr><td>$x-1=0$
$x=1$
$(1,0)$</td><td>$f(0)=\dfrac{0-1}{0^2-0-6}$
$f(0)=\dfrac{-1}{-6}$
$f(0)=\dfrac{1}{6}$
$\left(0,\dfrac{1}{6}\right)$</td><td>Find the intercepts.</td></tr>
<tr><td>$(x-3)(x+2)=0$
$x-3=0$ | $x+2=0$
$x=3$ | $x=-2$</td><td>$y=0$</td><td>Find the asymptotes.
The degree of the numerator (1) is less than the degree of the denominator (2) so the horizontal asymptote is $y=0$.</td></tr>
<tr><td colspan="2">
</td><td>The sign array determines whether the graph is positive or negative on the interval.</td></tr>
</table>

Intercepts	**Asymptotes**	**Final Sketch**
		

Domain: $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$ Range: $(-\infty,0)\cup(0,\infty)$	The domain includes all values except for the vertical asymptotes. The range includes all values except for the horizontal asymptote.

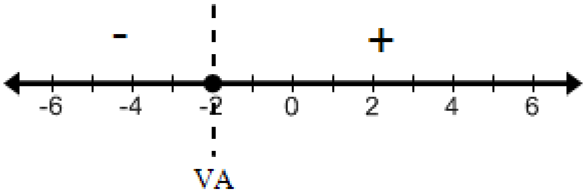
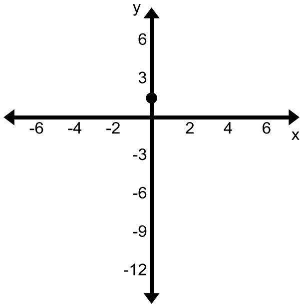
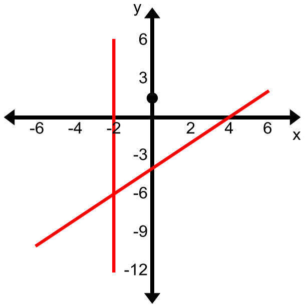
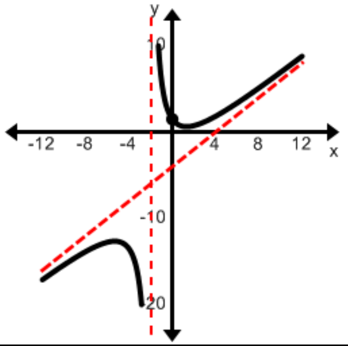
Example 2:

Graph the function $f\left(x\right)=\dfrac{2x^2-2}{x^2-4}$ and identify the domain and range.

<table>
<tr><td colspan="2">$f\left(x\right)=\dfrac{2\left(x-1\right)\left(x+1\right)}{\left(x-2\right)\left(x+2\right)}$</td><td>Factor the numerator and the denominator.</td></tr>
<tr><td>$2\left(x-1\right)\left(x+1\right)=0$
$x-1=0 \quad \big| \quad x+1=0$
$x=1 \quad \big| \quad x=-1$
$\left(1,0\right)$ or $\left(-1,0\right)$</td><td>$f\left(0\right)=\dfrac{2\left(0\right)^2-2}{\left(0\right)^2-4}$
$f\left(0\right)=\dfrac{0-2}{0-4}$
$f\left(0\right)=\dfrac{2}{4}=\dfrac{1}{2}$
$\left(0,\dfrac{1}{2}\right)$</td><td>Find the intercepts.</td></tr>
<tr><td>$\left(x-2\right)\left(x+2\right)=0$
$x-2=0 \quad \big| \quad x+2=0$
$x=2 \quad \big| \quad x=-2$</td><td>$y=\dfrac{2}{1}=2$</td><td>Find the asymptotes.
The degree of the numerator (2) is the same as the degree of the denominator (2) so the horizontal asymptote is the ratio of the leading coefficients.</td></tr>
<tr><td colspan="2"> </td><td>The sign array determines whether the graph is positive or negative on the interval.</td></tr>
<tr><td>**Intercepts**</td><td>**Asymptotes**</td><td>**Final Sketch**</td></tr>
<tr><td> </td><td> </td><td> </td></tr>
<tr><td colspan="2">Domain: $\left(-\infty,-2\right)\cup\left(-2,2\right)\cup\left(2,\infty\right)$
Range: $\left(-\infty,2\right)\cup\left(2,\infty\right)$</td><td>The domain includes all values except for the vertical asymptotes.
The range includes all values except for the horizontal asymptote.</td></tr>
</table>

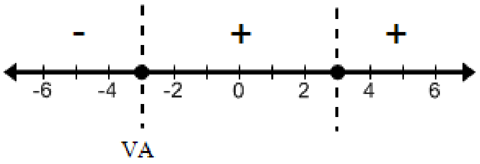
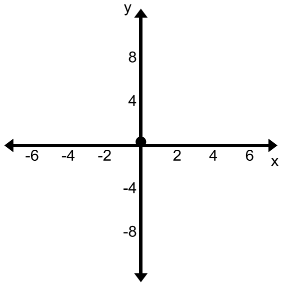
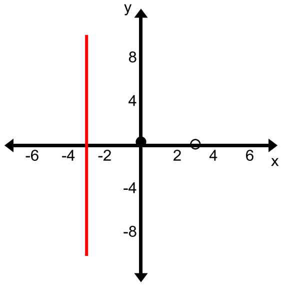
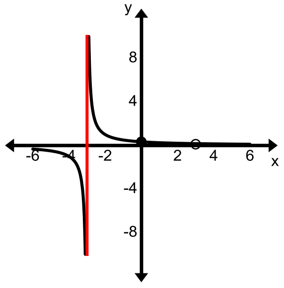
Example 3:

Graph the function $f(x)=\dfrac{x^2-2x+3}{x+2}$ and identify the domain.

<table>
<tr><td colspan="2">$f(x)=\dfrac{x^2-2x+3}{x+2}$</td><td>Factor the numerator and the denominator. The numerator does not factor.</td></tr>
<tr><td>$x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(3)}}{2(1)}$
$x=\dfrac{2\pm\sqrt{4-12}}{2}$
$x=\dfrac{2\pm\sqrt{-8}}{2}$
The solutions are not real.</td><td>$f(0)=\dfrac{(0)^2-2(0)+3}{(0)+2}$
$f(0)=\dfrac{3}{2}$
$\left(0,\dfrac{3}{2}\right)$</td><td>Find the intercepts.
Use the quadratic formula to find the x-intercepts.</td></tr>
<tr><td>$x+2=0$
$x=-2$</td><td>$$\begin{array}{r} x-4 \\ x+2\overline{)\,x^2-2x+3} \\ \underline{-(x^2+2x)} \\ -4x+3 \\ \underline{-(-4x-8)} \\ 11 \end{array}$$
$y=x-4$</td><td>Find the asymptotes.
The degree of the numerator (2) is greater than the degree of the denominator (1) so there may be an oblique asymptote.
The quotient is the oblique asymptote, the remainder is disregarded.</td></tr>
<tr><td colspan="2"> </td><td>The sign array determines whether the graph is positive or negative on the interval.</td></tr>
<tr><td>Intercepts</td><td>Asymptotes</td><td>Final Sketch</td></tr>
<tr><td> </td><td> </td><td> </td></tr>
<tr><td colspan="2">Domain: $(-\infty,-2)\cup(-2,\infty)$</td><td>The domain includes all values except for the vertical asymptotes.</td></tr>
</table>

Example 4:

Graph the function $f\left(x\right)=\frac{x-3}{x^2-9}$ and identify the domain and range.

<table>
<tr><td colspan="2">$f\left(x\right)=\frac{x-3}{\left(x-3\right)\left(x+3\right)}$</td><td colspan="2">Factor the numerator and the denominator.</td></tr>
<tr><td>$x-3=0$
$x=3$
$\left(3,0\right)$</td><td>$f\left(0\right)=\frac{0-3}{0^2-9}$
$f\left(0\right)=\frac{-3}{-9}$
$f\left(0\right)=\frac{1}{3}$
$\left(0,\frac{1}{3}\right)$</td><td colspan="2">Find the intercepts.</td></tr>
<tr><td>$\left(x-3\right)\left(x+3\right)=0$
$x-3=0$ | $x+3=0$
$x=3$ | $x=-3$</td><td>$y=0$</td><td colspan="2">Find the asymptotes.
Note: Only $x=-3$ is a vertical asymptote because the numerator and the denominator have a common factor. The function is still undefined at $x=3$ because it makes the denominator of the original function zero.
The degree of the numerator (1) is less than the degree of the denominator (2) so the horizontal asymptote is $y=0$.</td></tr>
<tr><td colspan="2">
</td><td colspan="2">The sign array determines whether the graph is positive or negative on the interval.</td></tr>
<tr><td>Intercepts</td><td colspan="2">Asymptotes</td><td>Final Sketch</td></tr>
<tr><td>
</td><td colspan="2">
</td><td>
</td></tr>
<tr><td colspan="2">Domain: $\left(-\infty,-3\right)\cup\left(-3,3\right)\cup\left(3,\infty\right)$
Range: $\left(-\infty,0\right)\cup\left(0,\infty\right)$</td><td colspan="2">The domain includes all values except for the values that make the denominator zero.
The range includes all values except for the horizontal asymptote.</td></tr>
</table>

Practice Exercises A

Graph each rational function and determine the domain and range.

1. $f\left(x\right)=\dfrac{3}{x+7}$
2. $f\left(x\right)=\dfrac{-6}{x+2}$
3. $f\left(x\right)=\dfrac{x-5}{x^2-16}$
4. $f\left(x\right)=\dfrac{x-1}{x^2-9}$
5. $f\left(x\right)=\dfrac{x+7}{x^2-3x-18}$
6. $f\left(x\right)=\dfrac{x-2}{x+1}$
7. $f\left(x\right)=\dfrac{-4x+1}{x-2}$
8. $f\left(x\right)=\dfrac{3x+2}{5x-6}$
9. $f\left(x\right)=\dfrac{x^2+x-30}{3x^2-3}$
10. $f\left(x\right)=\dfrac{2x^2+17x+21}{3x^2+4x-4}$
11. $f\left(x\right)=\dfrac{2x^2-9x+4}{x^2-7x+10}$
12. $f\left(x\right)=\dfrac{x-7}{x^2-4x-21}$
13. $f\left(x\right)=\dfrac{x+4}{x^2-2x-24}$
14. $f\left(x\right)=\dfrac{2x}{x^2+3x}$
15. $f\left(x\right)=\dfrac{x-3}{2x^2-x-15}$

Graph each rational function and determine the domain.

16. $f\left(x\right)=\dfrac{-2}{x^2-6x+8}$
17. $f\left(x\right)=\dfrac{3}{x^2-1}$
18. $f\left(x\right)=\dfrac{3x^2-x}{x+1}$
19. $f\left(x\right)=\dfrac{x^2+x-42}{x-4}$
20. $f\left(x\right)=\dfrac{x^3+8}{x^2-4}$
21. $f\left(x\right)=\dfrac{x^3+1}{x-1}$

Unit 4 Clusters 2 (F.IF.4, and F.IF.5): Key Features of Graphs

Cluster 2: Key features of graphs

4.2 Interpret key features (intercepts, intervals of increasing/decreasing, intervals of positive/negative, relative maximums or minimums, symmetries, and end behavior) of graphs and tables in terms of the quantities.

4.2 Sketch graphs showing key features given a verbal description of the relationship.

4.2 Relate the domain of a function to its graph and the relationship it describes.

VOCABULARY

The ***x*-intercept** is where a graph crosses or touches the x-axis. It is the ordered pair $(a,0)$. Where *a* is a real number.

The ***y* -intercept** is where a graph crosses or touches the y-axis. It is the ordered pair $(0,b)$. Where *b* is a real number.

A **relative maximum** occurs when the *y*-value is greater than all of the *y*-values near it. A function may have more than one relative maximum value. A **relative minimum** occurs when the *y*-value is less than all of the *y*-values near it. A function may have more than one relative minimum value.

An **interval** is a set of numbers between two x-values. An **open interval** is a set of numbers between two x-values that does not include the two end values. **Open intervals** are written in the form (x_1, x_2) or $x_1 < x < x_2$. A **closed interval** is a set of numbers between two x-values that does include the two end values. **Closed intervals** are written in the form $[x_1, x_2]$ or $x_1 \le x \le x_2$.

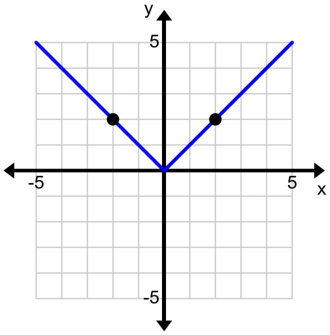
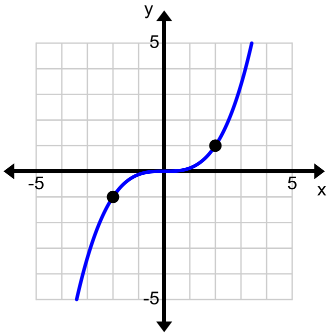
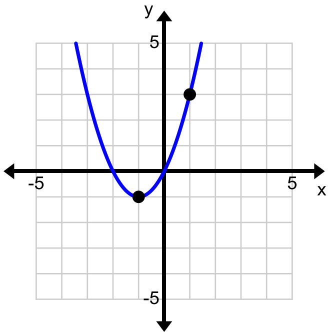
A function *f* is **increasing** when it is rising (or going up) from left to right and it is **decreasing** when it is falling (or going down) from left to right. A **constant** function is neither increasing nor decreasing; it has the same *y*-value for its entire domain.

A function is **positive** when $f(x) > 0$ or the *y*-coordinates are always positive. A function is **negative** when $f(x) < 0$ or the *y*-coordinates are always negative.

End behavior describes what is happening to the *y*-values of a graph when *x* goes to the far right $(+\infty)$ or *x* goes the far left $(-\infty)$.

End behavior is written in the following format:

Right End Behavior:	**Left End Behavior:**
$\lim_{x\to\infty} f(x) = c$	$\lim_{x\to-\infty} f(x) = c$

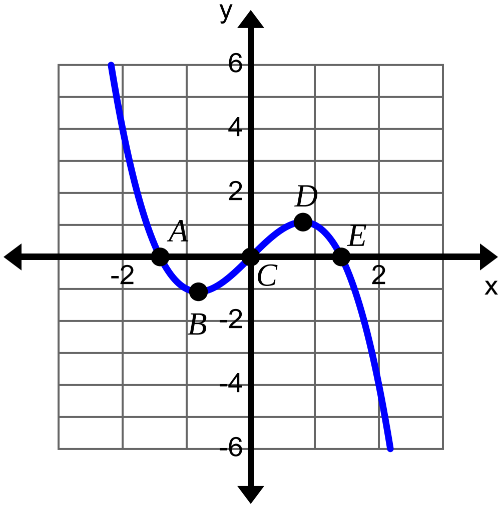
VOCABULARY	GRAPHICALLY	ALGEBRAICALLY
A function is **symmetric with respect to the *y*-axis** if, for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph. In other words, if you substitute –*x* in for every *x* you end up with the original function. When looking at the graph, you could "fold" the graph along the *y*-axis and both sides are the same.		$f(x)=\|x\|+5$ $f(-x)=\|-x\|+5$ $f(x)=f(-x)=\|x\|+5$
A function is **symmetric with respect to the origin** if, for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. In other words, if you substitute –*x* in for every *x* you end up with the opposite of the original function. When looking at the graph, there is a mirror image in Quadrants 1 & 3 or Quadrants 2 & 4.		$f(x)=8x^3$ $f(-x)=8(-x)^3$ $f(-x)=-f(x)=-8x^3$
An equation with **no symmetry**. If you substitute –*x* in for every *x* you end up with something that is neither the original function nor its opposite. When looking at the graph, you could not "fold" the graph along the *y*-axis and have both sides the same. It also does not reflect a mirror image in opposite quadrants.		$f(x)=x^2+2x$ $f(-x)=(-x)^2+2(-x)$ $f(-x)=x^2-2x\neq f(x)\neq -f(x)$

Example 1:

Analyze the key features of $f\left(x\right)=-x^3+2x$.

- a. Graph the function
- b. Identify the intercepts
- c. Identify the relative maximums and minimums
- d. Identify the intervals where the function is increasing or decreasing
- e. Identify the intervals where the function is positive or negative
- f. Determine the end behavior
- g. Determine the symmetry

a.



b. intercepts: A $\left(-1.414,0\right)$, C $\left(0,0\right)$, E $\left(1.414,0\right)$

c. maximum: D $\left(0.816,1.089\right)$; minimum B$\left(-0.816,-1.089\right)$

d. increasing: $\left(-0.816,0.816\right)$; decreasing $\left(-\infty,-0.816\right)\cup\left(0.816,\infty\right)$

e. positive: $\left(-\infty,-1.414\right)\cup\left(0,1.414\right)$; negative: $\left(-1.414,0\right)\cup\left(1.414,\infty\right)$

f. right end behavior: $\lim_{x\to\infty} f\left(x\right)=-\infty$; left end behavior: $\lim_{x\to-\infty} f\left(x\right)=\infty$

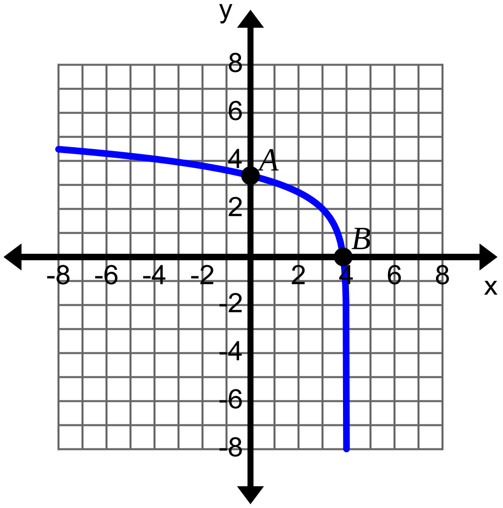
g. odd symmetry

Example 2:

Analyze the key features of $f\left(x\right)=\ln\left(4-x\right)+2$.

- a. Graph the function
- b. Identify the intercepts
- c. Identify the relative maximums and minimums
- d. Identify the intervals where the function is increasing or decreasing
- e. Identify the intervals where the function is positive or negative
- f. Determine the end behavior
- g. Determine the symmetry

a.



b. x-intercept: B $(3.865, 0)$;

y-intercept: A$(0, 3.386)$

c. maximum: none

d. increasing: decreasing$(-\infty, 4)$

e. positive: $(-\infty, 3.865)$; negative: $(3.865, 4)$

f. right end behavior: $\lim_{x \to 4^-} f(x) = -\infty$; left end behavior: $\lim_{x \to -\infty} f(x) = \infty$

g. no symmetry

Practice exercises A

Analyze the key features of $f(x)$.

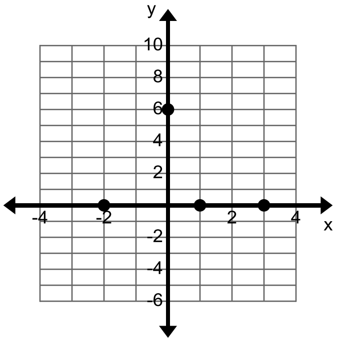
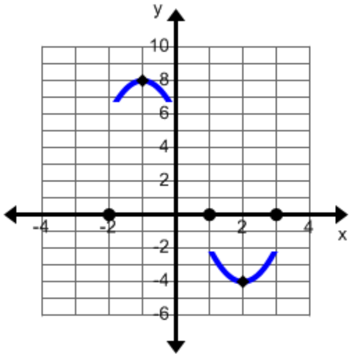
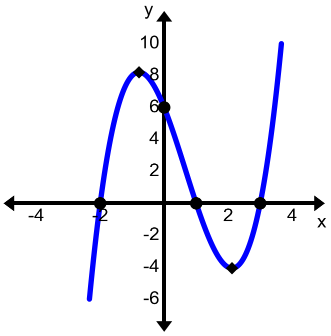
a. Graph the function
b. Identify the intercepts
c. Identify the relative maximums and minimums
d. Identify the intervals where the function is increasing or decreasing
e. Identify the intervals where the function is positive or negative
f. Determine the end behavior
g. Determine the symmetry

1. $f(x) = \frac{1}{2}|x-3|-5$
2. $f(x) = -2x^2+4$
3. $f(x) = \sqrt[3]{x+2}-3$
4. $f(x) = x^3+x^2-9x-9$
5. $f(x) = -3\sqrt{5-x}+2$
6. $f(x) = e^{x+4}-3$
7. $f(x) = \ln(x-3)-1$
8. $f(x) = 3\sin(-x)$
9. $f(x) = \cos(2x)$
10. $f(x) = \begin{cases} \frac{1}{2}x & \text{if } x < 2 \\ (x-4)^2-3 & \text{if } x \geq 2 \end{cases}$
11. $f(x) = -\frac{2}{3}x$
12. $f(x) = \text{int}\ \ x+2\ \ +3$

Example 3:
Use the characteristics to sketch a graph of the function described.

A polynomial function has:

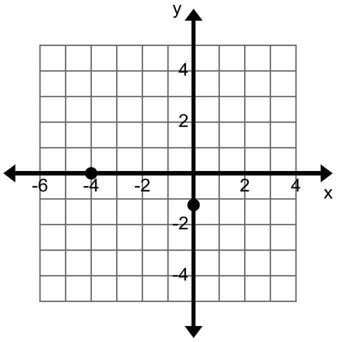
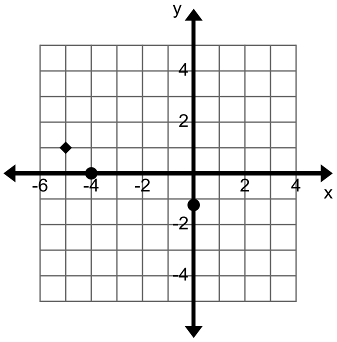
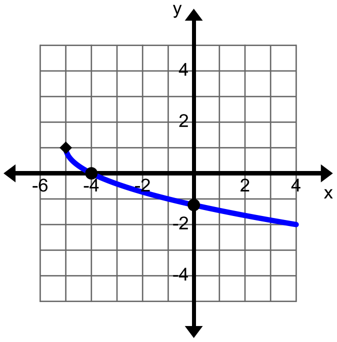
a. x-intercepts: $(-2,0)$, $(1,0)$, and $(3,0)$; y-intercept: $(0,6)$
b. relative maximum: $(-1,8)$; relative minimum: $(2,-4)$
c. right end behavior: $\lim\limits_{x\to\infty} f(x)=\infty$; left end behavior: $\lim\limits_{x\to-\infty} f(x)=-\infty$

a. Plot the intercepts.  	b. Plot the maximum and minimums  	c. Draw the end behavior in and connect the points. 

Example 4:
Use the characteristics to sketch a graph of the function described.

A square root function has:

a. x-intercept: $(-4,0)$; y-intercept: $(0,-1.236)$
b. relative maximum: $(-5,1)$
c. right end behavior: $\lim\limits_{x\to\infty} f(x)=-\infty$; left end behavior: $\lim\limits_{x\to-5^{+}} f(x)=1$

a. Plot the intercepts.  	b. Plot the maximum.  	c. Draw the end behavior in and connect the points. 

Practice Exercises B

Use the characteristics to sketch a graph of the function described.

1. $f(x)$ is an even function that decreases at a constant rate from $(-\infty, 0)$, has a minimum at $(0,-3)$ and a point at $(2,0)$.

2. $g(x)$ is a periodic, odd function that has x-intercepts at $(0,0)$ and $(\pm 4k, 0)$, a maximum at $(2,3)$ and a minimum at $(6,-3)$.

3. $h(x)$ is a function with a maximum value at $(1,9)$, intercepts at $(-2,0)$ and $(4,0)$, and end behavior $\lim\limits_{x\to\infty} h(x) = -\infty$ and $\lim\limits_{x\to-\infty} h(x) = -\infty$.

4. $j(x)$ is a function with an intercepts at $(-1,0)$ and $(0,3)$, end behavior $\lim\limits_{x\to-2^+} j(x) = -\infty$ and $\lim\limits_{x\to\infty} h(x) = \infty$, and passes through the point $(3,7)$.

5. $f(x)$ is a function that increases $(-\infty, \infty)$ with intercepts at $(4,0)$ and $(0,-6)$, and end behavior $\lim\limits_{x\to\infty} h(x) = \infty$ and $\lim\limits_{x\to-\infty} h(x) = -9$.

6. $g(x)$ is an even function with a minimum at $(-2,-5)$, a maximum at $(0,0)$, intercepts at $(-3,0)$ and $(3,0)$, and end behavior $\lim\limits_{x\to\infty} h(x) = \infty$.

7. $h(x)$ is a function with intercepts at $(-3,0)$, $(1,0)$, and $(0,-3)$, minimum at $(-1,-4)$ and maximums at $(-3,0)$ and $(1,0)$, and end behavior $\lim\limits_{x\to\infty} h(x) = -\infty$ and $\lim\limits_{x\to-\infty} h(x) = -\infty$.

8. $j(x)$ is a function with intercepts at $(-5,0)$, $(-2,0)$, $(0,0)$, $(2,0)$, and $(4,0)$, maximums at $(-4,38)$ and $(1,5)$, minimums at $(-1,-6)$ and $(3,-12)$, and end behavior $\lim\limits_{x\to\infty} h(x) = \infty$ and $\lim\limits_{x\to-\infty} h(x) = -\infty$.

Domain

VOCABULARY

The **domain** is the set of all first coordinates when given a table or a set of ordered pairs. It is the set of all *x*-coordinates of the points on the graph and is the set of all numbers for which a function is defined. The domain it is written from the least value to the greatest value.

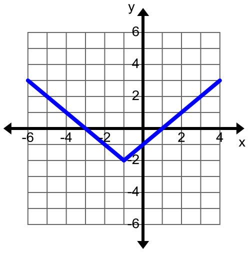
Example 5:

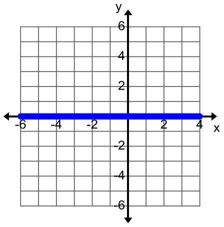
Find the domain of $f\left(x\right)=2\sqrt{x+2}-3$.

1. Find any values for which the function is undefined.	The square root function has real number solutions if the expression under the radicand is positive or zero. This means that $x+2\geq 0$ therefore $x\geq -2$.
2. Write the domain in interval notation.	The domain is $[-2,\infty)$.

Example 6:

Find the domain of the function graphed to the right.



1. List all the *x*-values of the function graphed.	If you were to flatten the function against the *x*-axis you would see something like this:  The function is defined for all the *x*-values along the *x*-axis.
2. Write the domain in interval notation.	The domain is $\left(-\infty,\infty\right)$.

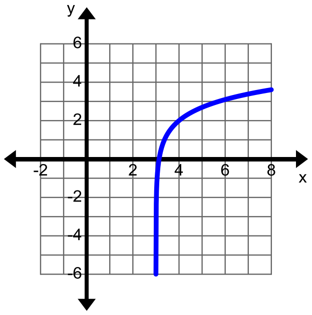
Example 7:

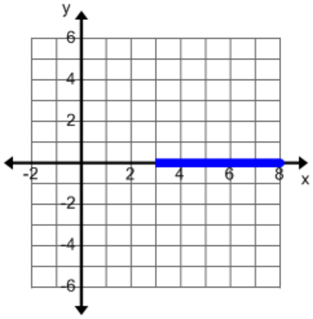
Find the domain of $f(x)=\dfrac{x+4}{x^2-3x-4}$.

1. Find any values for which the function is undefined. $x^2-3x-4=0$ $(x-4)(x+1)=0$ $x-4=0$ or $x+1=0$ $x=4$ $x=-1$	Division by zero is undefined, so exclude from the domain any values that would make the denominator zero.
2. Write the domain in interval notation.	The domain is $(-\infty,-1)\cup(-1,4)\cup(4,\infty)$.

Example 8:

Find the domain of $f(x)=\ln(x-3)+2$ graphed to the right.



1. List all the *x*-values of the function graphed.	If you were to flatten the function against the *x*-axis you would see something like this:  Notice that the vertical asymptote has been shifted to $x=3$. The function is defined for all the *x*-values along the *x*-axis from 3 to infinity.
2. Write the domain in interval notation.	The domain is $(3,\infty)$.

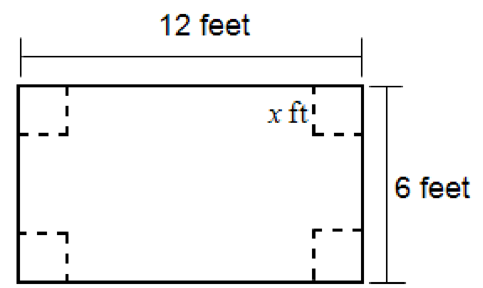
Example 9:

The path of a ball thrown straight up can be modeled by the equation $h(t) = -16t^2 + 48t + 12$ where *t* is the time in seconds that the ball is in the air and *h* is the height of the ball. What is the real world domain for the situation?

1. Find all the values that would make sense for the situation.	The domain represents the amount of time that the ball is in the air. At $t = 0$ the ball is thrown and enters the air shortly afterwards so the domain must be greater than zero. The ball will hit the ground at 3.232 seconds. Once it is on the ground it is no longer in the air so the domain must be less than 3.232 seconds. The ball is in the air for $0 < t < 3.232$ seconds.
2. Write the domain in interval notation.	The domain is $(0, 3.232)$.

Example 10:

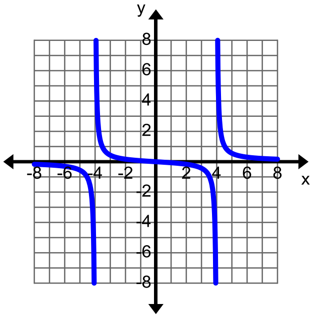
A square of side *x* feet is cut out of each corner of a 12 feet by 6 feet piece of material to form an open-topped box. Determine the domain of the volume function in terms of *x*.

1. Write a function for the volume in terms of *x*. The height is $h = x$. The length is $l = 12 - 2x$. The width is $w = 6 - 2x$. The volume is $V = lwh$. $V(x) = x(12-2x)(6-2x)$	
2. Find all the values that would make sense for the situation. $6 - 2x = 0$ $6 = 2x$ $3 = x$	The length of a side cannot be zero. Therefore, take the shorter side and set it equal to zero. The domain must be values that are less than this value but greater than zero.
3. Write the domain in interval notation.	The domain is $(0, 3)$.

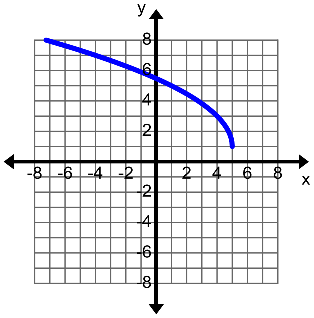
Practice Exercises C

Find the domain.

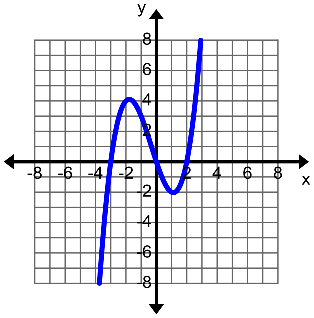
1.



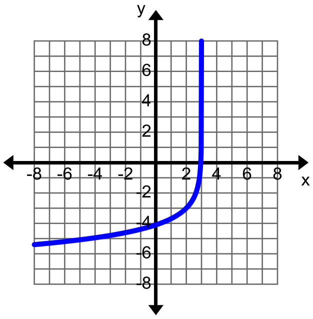
2.



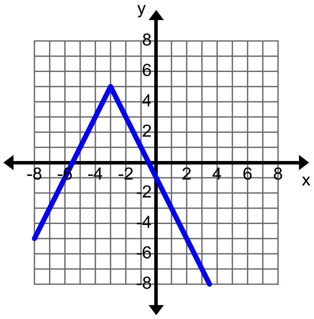
3.



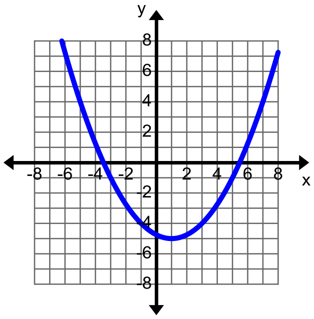
4.



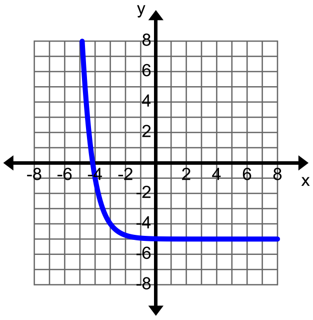
5.



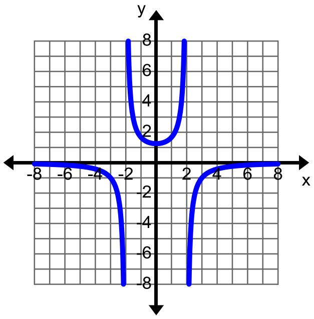
6.



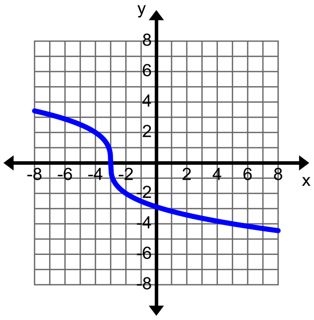
7.



8.



9.



10. $f(x)=1-2\sqrt{3-x}$

11. $f(x)=\frac{2x+1}{x^2+3x-10}$

12. $f(x)=-2\log(x+2)+1$

13. $f(x)=4\sqrt[3]{-\frac{1}{2}x}$

14. $f(x)=(x-2)^3-4$

15. $f(x)=\sin(\pi x)+3$

16. $f(x)=\frac{8x-3}{4x+5}$

17. $f(x)=2|x+4|-5$

18. $f(x)=\sqrt{x-4}$

19. A square of side x inches is cut out of each corner of a 30 inches by 24 inches piece of material to form an open-topped box. Determine the domain of the volume function in terms of x.

20. An object is dropped from the ledge of an open window that is 25 feet above the ground. What is the domain for the situation? (Use $h(t) = -16t^2 + 25$.)

21. A brick border of x feet is installed inside a garden that currently measures 20 feet by 15 feet. Determine the domain of the area function that describes the reduced garden in terms of x.

22. A parking garage charges $1.50 per hour with a maximum charge of $12 per day. Determine the domain.

Unit 4 Cluster 3 (F.IF.9): Comparing Functions

Cluster 3: Analyze Functions using Different Representations

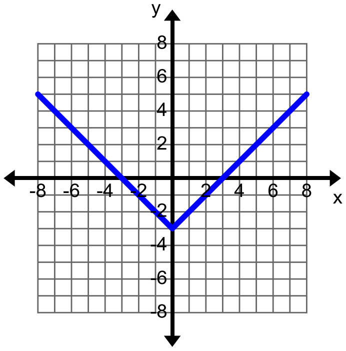
4.3 Compare properties (key features) of functions each represented differently (table, graph, equation or description)

Example 1:

Compare the properties of function A to those of function B.

a. x and y-intercepts
b. intervals of increasing or decreasing
c. intervals of positive or negative
d. maximums or minimums
e. domain and range
f. end behavior

Function A



Function B

$$y = x - 3$$

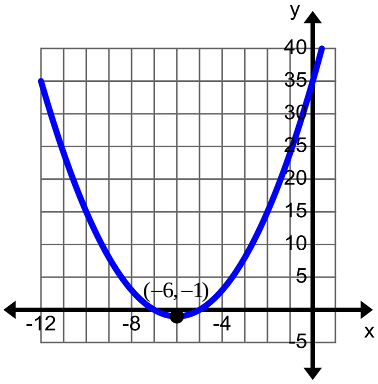
Function A	Function B
a. x-intercepts $(-3,0)$ and $(3,0)$; y-intercept $(0,-3)$	a. x-intercept $(3,0)$; y-intercept $(0,-3)$
b. increasing $(0,\infty)$; decreasing $(-\infty,0)$	b. increasing $(-\infty,\infty)$
c. positive $(-\infty,-3)\cup(3,\infty)$; negative $(-3,3)$	c. positive $(3,\infty)$; negative $(-\infty,3)$
d. minimum $(0,-3)$	d. none
e. domain $(-\infty,\infty)$; range $[-3,\infty)$	e. domain $(-\infty,\infty)$; range $(-\infty,\infty)$
f. $\lim\limits_{x\to\infty} f(x)=\infty$; $\lim\limits_{x\to-\infty} f(x)=\infty$	f. $\lim\limits_{x\to\infty} f(x)=\infty$; $\lim\limits_{x\to-\infty} f(x)=-\infty$

On the interval $(0,\infty)$ both functions have the same characteristics. However, on the interval $(-\infty,0)$ there are quite a few differences: function A has an additional x-intercept, function A has a minimum, function A is decreasing on the interval, function A is positive on part of the interval, and the left end behavior is different.

Example 2:

Compare the properties of function A to those of function B.

a. x and y-intercepts
b. intervals of increasing or decreasing
c. intervals of positive or negative
d. maximums or minimums
e. domain and range

<table>
<tr><th>Function A</th><th>Function B</th></tr>
<tr><td>
<table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>0</td><td>35</td></tr>
<tr><td>3</td><td>8</td></tr>
<tr><td>4</td><td>3</td></tr>
<tr><td>5</td><td>0</td></tr>
<tr><td>6</td><td>-1</td></tr>
<tr><td>7</td><td>0</td></tr>
<tr><td>8</td><td>3</td></tr>
<tr><td>9</td><td>8</td></tr>
</table>
</td><td>


</td></tr>
</table>

Function A	Function B
a. x-intercepts $(5,0)$ and $(7,0)$; y-intercept $(0,35)$	a. x-intercepts $(-7,0)$ and $(-5,0)$; y-intercept $(0,35)$
b. increasing $(6,\infty)$; decreasing $(-\infty,6)$	b. increasing $(-6,\infty)$; decreasing $(-\infty,-6)$
c. positive $(-\infty,5)\cup(7,\infty)$; negative $(5,7)$	c. positive $(-\infty,-7)\cup(-5,\infty)$; negative $(-7,-5)$
d. minimum $(6,-1)$	d. minimum $(-6,-1)$
e. domain $(-\infty,\infty)$; range $[-1,\infty)$	e. domain $(-\infty,\infty)$; range $[-1,\infty)$

Both functions have the same y-intercept, minimum value, domain, range, and end behavior. The points forthe minimum and the x-intercepts are reflections of one another over the y-axis.

Example 3:

Compare the properties of function A to those of function B.

- a. intercepts
- b. intervals of increasing or decreasing
- c. intervals of positive or negative
- d. maximums or minimums
- e. symmetry

<table>
<tr><th>Function A</th><th>Function B</th></tr>
<tr>
<td>$f\left(x\right)=x^3-4x$</td>
<td>
<table>
<tr><th>x</th><th>$f\left(x\right)$</th></tr>
<tr><td>-3</td><td>-192</td></tr>
<tr><td>-2</td><td>-48</td></tr>
<tr><td>-1</td><td>0</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>2</td><td>48</td></tr>
<tr><td>3</td><td>192</td></tr>
</table>
Relative minimum $\left(\sqrt{\tfrac{1}{3}},-3.079\right)$
Relative Maximum $\left(-\sqrt{\tfrac{1}{3}},3.079\right)$
</td>
</tr>
</table>

Function A	Function B
a. intercepts $\left(-2,0\right)$, $\left(0,0\right)$ and $\left(2,0\right)$ b. increasing $\left(-\infty,-2\sqrt{\tfrac{1}{3}}\right)\cup\left(2\sqrt{\tfrac{1}{3}},\infty\right)$; decreasing $\left(-2\sqrt{\tfrac{1}{3}},2\sqrt{\tfrac{1}{3}}\right)$ c. positive $\left(-2,0\right)\cup\left(2,\infty\right)$; negative $\left(-\infty,-2\right)\cup\left(0,2\right)$ d. minimum $\left(2\sqrt{\tfrac{1}{3}},-3.079\right)$; maximum $\left(-2\sqrt{\tfrac{1}{3}},3.079\right)$ e. odd symmetry	a. intercepts $\left(-1,0\right)$, $\left(0,0\right)$ and $\left(1,0\right)$ b. increasing $\left(-\infty,-\sqrt{\tfrac{1}{3}}\right)\cup\left(\sqrt{\tfrac{1}{3}},\infty\right)$; decreasing $\left(-\sqrt{\tfrac{1}{3}},\sqrt{\tfrac{1}{3}}\right)$ c. positive $\left(-1,0\right)\cup\left(1,\infty\right)$; negative $\left(-\infty,-1\right)\cup\left(0,1\right)$ d. minimum $\left(\sqrt{\tfrac{1}{3}},-3.079\right)$; maximum $\left(-\sqrt{\tfrac{1}{3}},3.079\right)$ e. odd symmetry

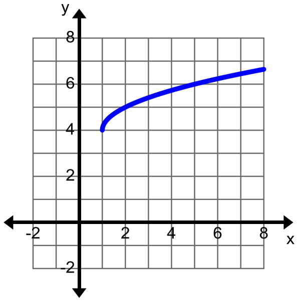
The functions have the same relative minimum and maximum value, but it occurs in different places. They both have the same *y*-intercept and they both have odd symmetry. It would seem that function A has twice the width as function B, but they seem to behave the same way between intercepts and maximums and minimums.

Practice Exercises A

1. Compare the properties of function A to those of function B.

   a. intervals of increasing or decreasing
   b. intervals of positive or negative
   c. maximums or minimums
   d. domain and range

Function A:



Function B:

$$y = -\sqrt{x-1}+4$$

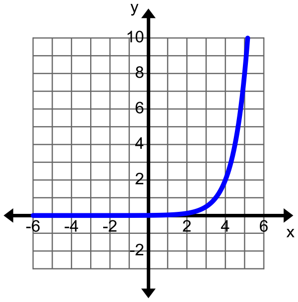
2. Compare the properties of function A to those of function B.

   a. y-intercept
   b. average rate of change on the interval $[4,5]$
   c. intervals of positive or negative
   d. domain and range

Function A:

x	$f(x)$
0	$\frac{1}{64}$
1	$\frac{1}{16}$
2	$\frac{1}{4}$
3	1
4	4
5	16

Function B:



3. Compare the properties of function A to those of function B.

   a. x-intercepts and y-intercepts
   b. maximums or minimums
   c. range
   d. symmetry

Function A:

x	$f(x)$
-6	0
-5	2
-3	4
-1	2
0	0
1	-2
3	-4
5	-2
6	0

Function B:

$$f(x) = 4\sin\left(\frac{\pi}{6}x\right)$$

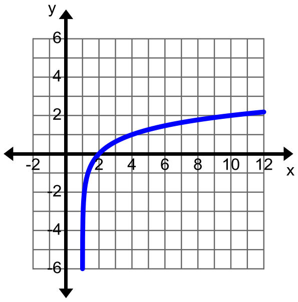
4. Compare the properties of function A to those of function B.

   a. x-intercept
   b. intervals of increasing or decreasing
   c. intervals of positive or negative
   d. domain

Function A:

x	$f(x)$
1	Undefined
$1.\overline{1}$	0
2	2
4	3
6	3.465
8	3.771
10	4

Function B:



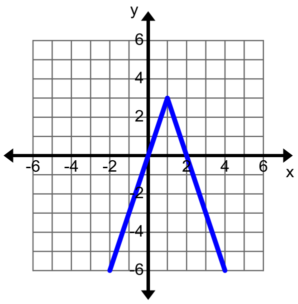
5. Compare the properties of function A to those of function B.

   a. *x*-intercepts and *y*-intercepts
   b. intervals of increasing and decreasing
   c. maximums or minimums
   d. domain and range

Function A:

$$f(x) = -3|x-1| + 6$$

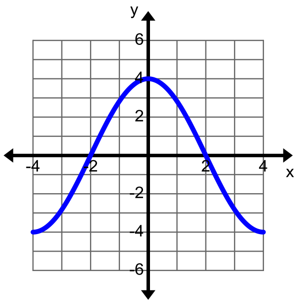
Function B:



6. Compare the properties of function A to those of function B.

   a. *x*-intercepts and *y*-intercepts
   b. maximums or minimums
   c. range
   d. symmetry

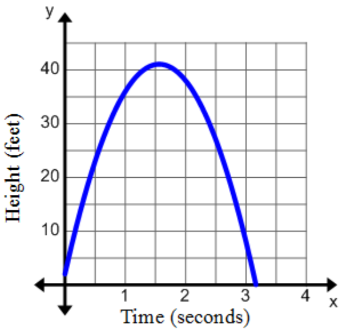
Function A:



Function B:

x	$f(x)$
-2	-4
-1	0
0	4
1	0
2	-4

7. A water powered rocket can be launched from a 0.16 foot platform straight up into the air with an initial velocity of 47 feet per second. A similar water powered rocket's height has been graphed at the right. Which rocket is in the air the longest? Which rocket has the greatest maximum height?



8. Compare the properties of function A to those of function B.
   a. x and y-intercepts
   b. intervals of increasing or decreasing
   c. intervals of positive or negative
   d. maximums or minimums

Function A:

$$f\left(x\right)=(x-2)^3-1$$

Function B:

x	$f\left(x\right)$
-4	-9
-3	-2
-2	-1
-1	0
0	7
1	26
2	63

Unit 4 Cluster 2(F.IF.6) Average Rate of Change

Cluster 2: Interpret Functions that Arise in Applications in Terms of a Context

4.2 Calculate, interpret, and estimate from a graph the average rate of change over an interval. Include rational, square root, cube root, polynomial, logarithmic, and trigonometric functions in addition to quadratic and exponential.

VOCABULARY

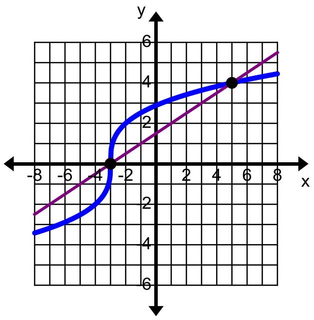
The **average rate of change** of a function over an interval is the ratio of the difference (change) in *y* over the difference (change) in *x*.

$$\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The line connecting the two points is called the **secant line**.

Example 1:

Find the average rate of change for $f(x) = 2\sqrt[3]{x+3}$ on the interval $[-3, 5]$.

$f(-3) = 2\sqrt[3]{-3+3}$ $f(-3) = 2\sqrt[3]{0}$ $f(-3) = 0$	$f(5) = 2\sqrt[3]{5+3}$ $f(5) = 2\sqrt[3]{8}$ $f(5) = 4$	First, find the value of the function at each end point of the interval.
 $m = \frac{4-0}{5-(-3)} = \frac{4}{8} = \frac{1}{2}$		Next, find the slope between the two points $(-3, 0)$ and $(5, 4)$.
The average rate of change of $f(x) = 2\sqrt[3]{x+3}$ on the interval $[-3, 5]$ is $\frac{1}{2}$.		

Example 2:
The table shows the total US farm exports in billions for several years. Find the average amount per year from 1996 to 2000.

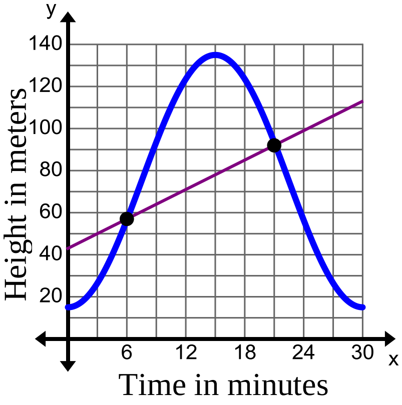
Years	Amount (billions)
1980	41.2
1985	29.0
1990	39.5
1992	43.2
1993	42.9
1994	46.3
1995	56.3
1996	60.4
1997	57.2
1998	51.8
1999	48.5
2000	51.6

The year 1996 is 16 years since 1980 and 2000 is 20 years since 1980, therefore the interval is [16, 20]. Find the slope between the two points (16, 60.4) and (20, 51.6).

$$m = \frac{51.6 - 60.4}{20 - 16} = \frac{-8.8}{4} = -2.2$$

The average rate of change from 1996 to 2000 is -2.2 billions of exports each year. This means that the number of exports decreases about 2.2 billion each year between 1996 and 2000.

Example 3:
Jane is visiting London and took a ride on the London Eye. Her distance in meters from the ground at any given time is shown in the graph at the right. Find her average rate of change from 6 to 21 minutes.



At 6 minutes her height appears to be close to 60 meters and at 21 minutes her height appears to be 90 meters. Find the slope between the two points (6, 60) and (21, 90).

$$m=\frac{90-60}{21-6}=\frac{30}{15}=2$$

Jane's average rate of change is 2 meters per minute. This means that she is traveling at an average rate of 2 meters per minute from 6 to 21 minutes.

Practice Exercises A

Find the average rate of change for each function on the specified interval.

1. $f(x)=3x^2-x+5$ on [-1, 3]
2. $f(x)=4x^2+12x+9$ on [-3, 0]
3. $f(x)=-x^2+4$ on $[3,a]$
4. $f(x)=\frac{x-7}{x^2+14x+40}$ on $[-9,-5]$
5. $f(x)=\frac{x^2+11x+30}{x+6}$ on $[-4,0]$
6. $f(x)=\frac{x^2+x-72}{x^2+5x}$ on $[-4,-1]$
7. $f(x)=\sqrt{x+8}-6$ on $[-4,1]$
8. $f(x)=\sqrt{x-7}+4$ on $[7,11]$
9. $f(x)=2\sqrt{x+3}-10$ on $[1,6]$
10. $f(x)=\sqrt[3]{x}-6$ on $[-1,1]$
11. $f(x)=\sqrt[3]{x+2}-1$ on $[-3,6]$
12. $f(x)=-\sqrt[3]{x+6}+2$ on $[-5,2]$
13. $f(x)=x^3+x-2$ on $[-3,2]$
14. $f(x)=x^4-8x^3+16x^2$ on $[-1,3]$
15. $f(x)=x^3-9x$ on $[-2,2]$
16. $f(x)=\log(x-3)-4$ on $[4,13]$
17. $f(x)=2\ln(x+1)-3$ on $[0,3]$
18. $f(x)=\ln(-x+2)-5$ on $[-7,1]$
19. $f(x)=4\sin x+7$ on $\left[-\pi,\frac{\pi}{2}\right]$
20. $f(x)=-2\cos x-3$ on $[0,\pi]$
21. $f(x)=3\sin x-3$ on $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$
22. $f(x)=4^{x-1}-5$ on $[-1,3]$
23. $f(x)=-2\cdot 3^x+4$ on $[1,3]$
24. $f(x)=\frac{1}{2}\left(\frac{1}{4}\right)^x+2$ on $[-5,-1]$

Find the average rate of change on the specified interval and interpret its meaning.

25. The average temperature per month is shown in the table below. Find the average rate of change from March to October.

Month	Temperature $^\circ F$
January	34
February	30
March	39
April	44
May	58
June	67
July	78
August	80
September	72
October	63
November	51
December	40

26. The table shows average annual consumption of cheese per person in the U.S. for selected years. (Source: U.S. Department of Agriculture). What is the average consumption between 1940 and 1995.

Year	Pounds Consumed
1910	4
1940	5
1970	8
1975	10
1995	25
2001	30

27. The table below shows the percentage of the U.S. labor force in unions for selected years between 1955 and 2005. Find the average rate of change from 1975 to 1995.

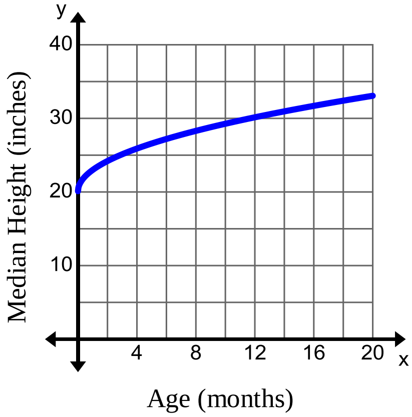
Year	Percent
1955	33.2
1960	31.4
1965	28.4
1970	27.3
1975	25.5
1980	21.9
1985	18.0
1990	16.1
1995	14.9
2000	13.5
2005	12.5

28. The table below shows the amount of carbon dioxide in the Earth's atmosphere for selected years. (Source: the Weather Channel.) Find the average rate of change from 1968 to 2003.

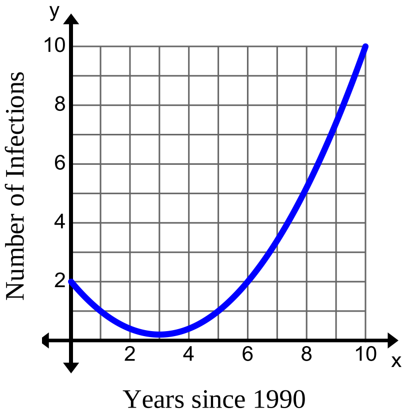
Year	CO_2 in Atmosphere (ppm)
1968	324.14
1983	343.91
1998	367.68
2003	376.68
2008	385.60

Find the average rate of change and interpret its meaning.

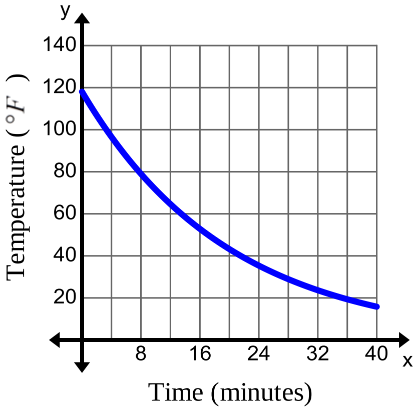
29. The graph below represents the height in inches of boys age *x* months. Find the average rate of change from 6 months to 16 months.



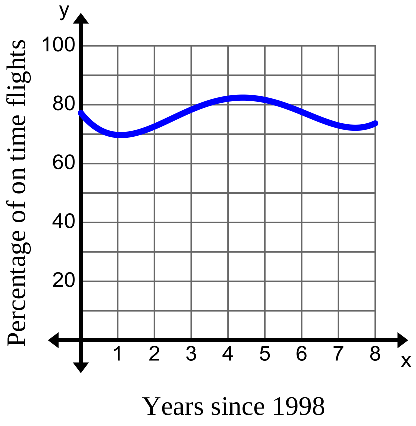
30. The graph below displays the number of infections per month for every 1,000 computers since 1990. Find the average rate of change from 1991 to 1998.



31. A cup of hot liquid is left out to cool. The graph below displays its temperature over time. Find the average rate of change from 4 to 12 minutes.



32. The graph below shows the percentage of on time flights per year since 1998. Find the average rate of change from 1999 to 2003.



Unit 2 Cluster 9 (A.REI.11): Solving Systems of Equations Graphically

Cluster 9: Represent and solve equations and inequalities graphically.

- 2.9 Explain why the *x*-coordinate of the points where the graphs intersect are solutions of the equation.
- 2.9 Find the solutions approximately using technology (linear, polynomial, rational, absolute value, exponential, and logarithmic functions).

Example 1:

Using the tables below, find when $f\left(x\right) \approx g\left(x\right)$ if $f\left(x\right) = x^2 - 3$ and $g\left(x\right) = -\frac{1}{2}x + 1$.

x	$f(x)$
-3	6
-2	1
-1	-2
0	-3
1	-2
2	1
3	6

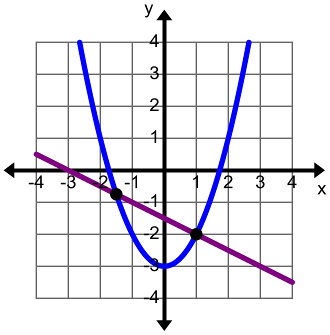
x	$g(x)$
-3	0
-2	-0.5
-1	-1
0	-1.5
1	-2
2	-2.5
3	-3

x	$f(x)$
-3	6
-2	1
-1	-2
0	-3
1	-2
2	1
3	6

x	g(x)
-3	0
-2	-0.5
-1	-1
0	-1.5
1	-2
2	-2.5
3	-3

It is obvious that the functions have the same *y*-value when $x = 1$, which means that the two functions intersect when $x = 1$.

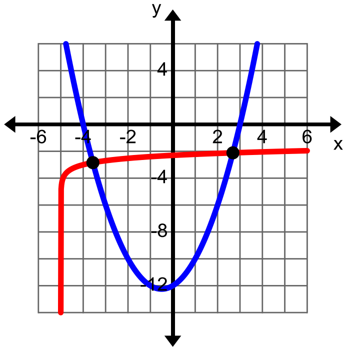
However, it is more difficult to see the second intersection. Notice that on the interval $-2 \le x \le -1$ the value of $f\left(x\right)$ is between 1 and -2. Similarly, on the interval $-2 \le x \le -1$ the value of $g\left(x\right)$ is between -0.5 and -1. Since both functions are continuous, they take on every value between 1 and -2 for $f\left(x\right)$ and between -0.5 and -1 for $g\left(x\right)$ which means that they will have the same *y*-value at some *x*-value between -2 and -1.



To find the second intersection more accurately, use technology to graph the functions and find the intersection. The intersection is $\left(-0.5, -0.75\right)$ so the functions are the same value when $x = -0.5$.

Example 2:

Use technology to find when $f(x)=g(x)$ if $f(x)=x^2+x-12$ and $g(x)=\log(x+5)-3$.



Graph both functions and find the intersection(s).

Note: There are two logarithmic buttons on your calculator. The common logarithm (log) which is base 10 and the natural logarithm (ln) which is base e.

$f(x)$ and $g(x)$ are the same value when $x=-3.567$ and $x=2.684$.

Practice Exercises A

Use the tables to find when $f(x)\approx g(x)$.

1. $f(x)=x^2+x+4$, $g(x)=2x+6$

x	$f(x)$
-3	10
-2	6
-1	4
0	4
1	6
2	10
3	16

x	g(x)
-3	0
-2	2
-1	4
0	6
1	8
2	10
3	12

2. $f(x)=3x^2+2x-18$, $g(x)=-\frac{1}{2}x-1$

x	$f(x)$
-4	22
-3	3
-2	-10
-1	-17
0	-18
1	-13
2	-2

x	g(x)
-4	1
-3	0.5
-2	0
-1	-0.5
0	-1
1	-1.5
2	-2

3. $f(x)=x^2-14$, $g(x)=|x+2|-10$

x	$f(x)$
-3	-5
-2	-10
-1	-13
0	-14
1	-13
2	-10
3	-5

x	g(x)
-3	-9
-2	-10
-1	-9
0	-8
1	-7
2	-6
3	-5

4. $f(x)=5x+2$, $g(x)=|x-4|-2$

x	$f(x)$
-3	-13
-2	-8
-1	-3
0	2
1	7
2	12
3	17

x	g(x)
-3	5
-2	4
-1	3
0	2
1	1
2	0
3	-1

Practice Exercises B

Use technology to find when $f(x)=g(x)$.

1. $f(x)=-2x+4$
   $g(x)=x^2+3$

2. $f(x)=\frac{1}{3}x-2$
   $g(x)=x^3-3x^2-4x$

3. $f(x)=-\frac{5}{2}x+3$
   $g(x)=\frac{x-1}{2x+1}$

4. $f(x)=-\frac{3}{4}x+7$
   $g(x)=-|x-5|+4$

5. $f(x)=4x-3$
   $g(x)=3^{x-2}-5$

6. $f(x)=-\frac{1}{10}x+3$
   $g(x)=\log(x+2)$

7. $f(x)=(x-5)^2-3$
   $g(x)=\frac{1}{5}x^3-\frac{12}{5}x^2+7x$

8. $f(x)=x^2+12x+31$
   $g(x)=\frac{x-4}{x^2-4x+4}$

9. $f(x)=(x-2)^2-4$
   $g(x)=|x-1|+3$

10. $f(x)=2x^2-7$
    $g(x)=5^{x-2}-3$

11. $f(x)=x^2-6x-7$
    $g(x)=\log(x+1)+3$

12. $f(x)=|2x-1|-3$
    $g(x)=\frac{1}{2}x^3-\frac{3}{2}x^2-\frac{13}{2}x+\frac{15}{2}$

13. $f(x)=\frac{2}{3}|x+1|-8$
    $g(x)=\frac{2x^2}{x^2-4x+45}$

14. $f(x)=|x+4|+3$
    $g(x)=3^{x-2}+1$

15. $f(x)=|x+6|$
    $g(x)=\log(5-x)+3$

Unit 4 Cluster 1 (A.CED.1, A.SSE.2, and A.CED.4): Writing and Solving Equations and Inequalities in One Variable

Cluster 1: Create Equations that describe numbers or relationships

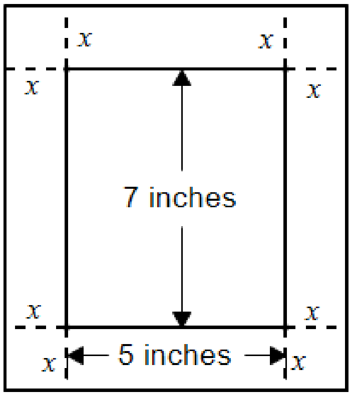
- 4.1 Create equations in one variable and use them to solve problems (include simple rational, square root, and polynomial)
- 4.1 Create inequalities in variable and use them to solve problems (include simple rational, square root, and polynomial)
- 4.1 Rearrange formulas to highlight a quantity of interest, using the same reasoning as solving equations.
- 2.2 Use the structure of an expression to rewrite it.

When solving contextual type problems it is important to:

- Identify what you know.
- Determine what you are trying to find.
- Draw a picture to help you visualize the situation when possible. Remember to label all parts of your drawing.
- Use familiar formulas to help you write equations.
- Check your answer for reasonableness and accuracy.
- Make sure you answered the entire question.
- Use appropriate units.

Example 1:

You want to create a custom border for a picture of you and your closest friends. The picture measures 5 inches by 7 inches. What should the width of the border be if the final area, including the border, is twice the area of the picture?



$A = lw$ $2(7 \cdot 5) = (7+2x)(5+2x)$	The area of a rectangle is the product of the length and width. The length is $l = 7+2x$. The width is $w = 5+2x$. You want the final area to be twice the area of the picture.
$70 = 4x^2 + 24x + 35$ $0 = 4x^2 + 24x - 35$	Simplify the expression and make sure the equation is equal to zero.
$x = 1.2130749$ The border should be about 1.2 inches wide.	Use the quadratic formula or technology to find the zero.

Example 2:
The height of a plastic rectangular prism storage container is 4 inches shorter than the width. The length is 7 inches longer than the width. The volume of the storage container is 5304 square inches. What are the dimensions of the container?

$h = w - 4$ $l = w + 7$	Let *w* represent the width of the box. Write equations for the height and the length in terms of the width.
$V = lwh$ $5304 = (w+7)(w)(w-4)$	The volume of a rectangular prism is $V = lwh$. Substitute in the values for the length , height, and volume of the box.
$5304 = w^3 + 3w^2 - 28w$	Expand the right side of the equation.
$0 = w^3 + 3w^2 - 28w - 5304$ $w = 17$	Make sure the equation is equal to zero then use technology to find the zeros.
The width is 17 inches, the length is 24 inches, and the height is 13 inches.	$h = 17 - 4 = 13$ $l = 17 + 7 = 24$

Work Problems

The equation $\frac{t}{a} + \frac{t}{b} = 1$, where *a* is the amount of time for A to complete the work alone, *b* is the amount of time for B to complete the work alone, and *t* is the amount of time needed for A and B to complete the work together, can be used to find the amount of time required for work to be done.

Example 3:
Britton can refinish the floor in 9 hours. Britton and Jason can refinish the floor together in 4 hours. How long would it take Jason to finish the floors himself?

$\frac{t}{a} + \frac{t}{b} = 1$	Use the work formula.
$\frac{4}{9} + \frac{4}{b} = 1$	Substitute the values you know. Let $a = 9$ and $t = 4$.
$\frac{4}{b} = \frac{5}{9}$ $36 = 5b$ $\frac{36}{5} = b$ $7.2 = b$	Solve for *b*.
It would take Jason 7 hours and 12 minutes to refinish the floors by himself.	0.2 of an hour (60 minutes) is 12 minutes. $0.2 \times 60 = 12$

Example 4:
A tugboat goes 12 mph in still water. It travels 45 miles upstream and 45 miles back in 8 hours. What is the speed of the current?

$\frac{d}{r_1}+\frac{d}{r_2}=t$ $\frac{45}{12-c}+\frac{45}{12+c}=8$	Recall that $d=rt$. We know the total time so solving this equation for time yields $\frac{d}{r}=t$. The total time for the trip was 8 hours. Traveling upstream, the boat moves against the current so the rate is $r=12-c$. Coming back the boat moves with the current so the rate is $r=12+c$.
$\left(\frac{45}{12-c}+\frac{45}{12+c}=8\right)\left(144-c^2\right)$ $45\left(12+c\right)+45\left(12-c\right)=1152-8c^2$ $540+45c+540-45c=1152-8c^2$ $1080=1152-8c^2$	Multiply by the LCM $\left(12-c\right)\left(12+c\right)=144-c^2$. Then simplify.
$-72=-8c^2$ $9=c^2$ $\pm 3=c$	Solve for *c*.
The speed of the current will not be negative so the current is 3 mph.	

Practice Exercises A

1. An open box is made from a rectangular piece of cardboard measuring 12 inches by 16 inches by cutting identical squares from the corners and turning up the sides. What are the lengths of the sides of the removed squares if the area of the bottom of the open box is 60 square inches?

2. A triangular table top has a base that is twice as long as its height. If the area of the table surface is 324 square inches, what is the value of the height and the base?

3. A family had three children and were expecting a fourth. The oldest was 3 years older than the youngest. The youngest was one year younger than the middle child. How old was each of the children on the day their new sibling was born if the product of their ages was 987 more than three times the sum of their ages.

4. The width of a box is two inches less than twice the height. The length is 4 inches less than three times the height. The volume is 2240 cubic inches. What are the dimensions of the box?

5. The junior class president and vice president have decided to call all of the junior class to remind them of junior pride week. The president, working alone, can call all of the juniors in six days. The vice president, working along, can call all of the juniors in four days. How long would it take to call of the juniors if they worked together?

6. Suzie can run 2.5 miles per hour faster than Jeff. In the time that it takes Suzie to run 6 miles, Jeff runs 4 miles. Find the speed of each runner.

7. Eva and Emily can clean the entire house in 4 hours. Eva can do it by herself in 6 hours. How long would it take Emily to do it by herself?

8. Sam can paddle a canoe in still water at a speed of 55 meters per minute. If he paddles upstream 135 meters in 3 minutes, what is the speed of the current?

9. The velocity of water flow, in feet per second, from a fire hose nozzle is given by $v(p) = 12.1\sqrt{p}$, where p is the nozzle pressure, in pounds per square inch (psi). Find the nozzle pressure if the water flow velocity is 110 feet per second. (Source: Houston Fire Department Continuing Education).

10. The frequency, in hertz, of a violin string can be modeled by the equation $f(t) = 49.1\sqrt{t}$, where t is the tension in newtons. What is the amount of tension applied if the frequency of the violin string is 278 hertz?

Using the Structure of Expressions to Solve Equations

Example 5:

Solve $\left(2x+5\right)^2-3\left(2x+5\right)-40=0$.

$\left(2x+5\right)^2-3\left(2x+5\right)-40=0$	
$u^2-3u-40=0$ $\left(u+5\right)\left(u-8\right)=0$ $u+5=0$ or $u-8=0$ $u=-5$ $u=8$	Let $u=2x+5$ and rewrite the equation in terms of *u*. Solve for *u*.
$2x+5=-5$ $2x+5=8$ $2x=-10$ or $2x=3$ $x=-5$ $x=\frac{3}{2}$	Substitute $2x+5$ in for *u* and solve for *x*.

Example 6:

Solve $\frac{1}{\left(2x-1\right)^2}+\frac{5}{2x-1}=-6$.

$\frac{1}{\left(2x-1\right)^2}+\frac{5}{2x-1}=-6$	
$u^2+5u=-6$ $u^2+5u+6=0$ $\left(u+3\right)\left(u+2\right)=0$ $u+3=0$ or $u+2=0$ $u=-3$ $u=-2$	Let $u=\frac{1}{2x-1}$ and rewrite the equation in terms of *u*. Solve for *u*.
$\frac{1}{2x-1}=-3$ $\frac{1}{2x-1}=-2$ $1=-3\left(2x-1\right)$ $1=-2\left(2x-1\right)$ $1=-6x+3$ $1=-4x+2$ $-2=-6x$ $-1=-4x$ $\frac{1}{3}=x$ $\frac{1}{4}=x$	Substitute $\frac{1}{2x-1}$ in for *u* and solve for *x*.

Example 7:

Solve $4x^3 = 8x^2$.

$4x^3 = 8x^2$	
$4x^3 - 8x^2 = 0$ $4x^2(x-2) = 0$	Collect all the terms on one side of the equation and factor the expression.
$4x^2 = 0$ $x^2 = 0$ or $x = 0$ $x - 2 = 0$ $x = 2$	Set each factor equal to zero and solve for *x*.

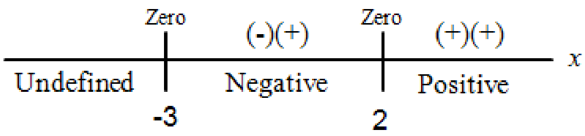
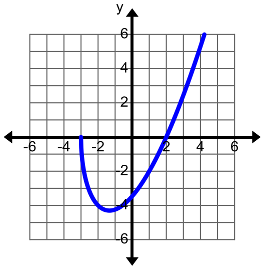
Practice Exercises B

1. $(x+3)^2 - 2(x+3) - 24 = 0$
2. $(x+1)^2 + 8(x+1) + 15 = 0$
3. $3(2-x)^2 + 5(2-x) + 2 = 0$
4. $x + \sqrt{x} = 12$
5. $x + 3\sqrt{x} = 4$
6. $x^{1/2} - 2x^{1/4} + 1 = 0$
7. $\dfrac{1}{(x+2)^2} = \dfrac{1}{x+2} + 2$
8. $\dfrac{1}{(x+5)^2} - \dfrac{4}{x+5} = 12$
9. $\left(\dfrac{x}{x+1}\right)^2 - \dfrac{2x}{x+1} = 8$
10. $x^{2/3} + 9x^{1/3} + 20 = 0$
11. $2x^{2/3} - 5x^{1/3} - 3 = 0$
12. $x^{4/3} - 6x^{2/3} + 9 = 0$
13. $3x^{4/3} + 5x^{2/3} - 2 = 0$
14. $x - 3x\sqrt{x} = 0$
15. $x - x\sqrt{x} = 0$
16. $x^5 + 4x^4 = 21x^3$
17. $x^3 = 16x$
18. $x^5 = 5x^3$
19. $x^3 - 3x^2 - 18x = 0$
20. $x^3 - 3x^2 - 4x + 12 = 0$
21. $x^3 - 3x^2 - x + 3 = 0$

Solving One Variable Inequalities

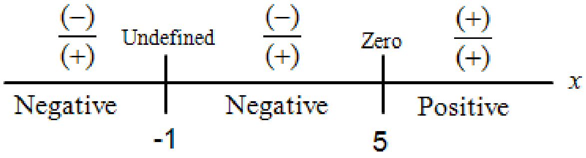
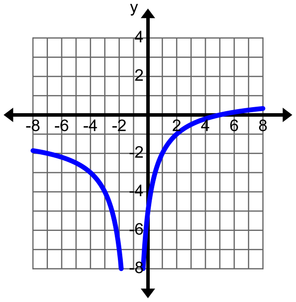
Example 8:

Solve $\left(x-2\right)\sqrt{x+3}\ge 0$.

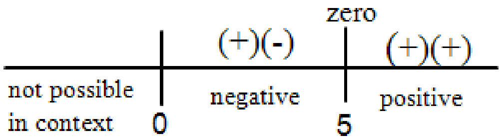
<table>
<tr><td>$\left(x-2\right)\sqrt{x+3}\ge 0$</td><td>The square root makes it so that the expression on the left side of the equation is undefined if $x<-3$. The left side of the equation is zero when $x=-3$ and $x=2$.</td></tr>
<tr><td> </td><td>Create a sign chart, using the zeros, to determine where the expression is positive or equal to zero.</td></tr>
<tr><td>The solution is $\{-3\}\cup[2,\infty)$.</td><td>
The graph confirms the solution.</td></tr>
</table>

Example 9:

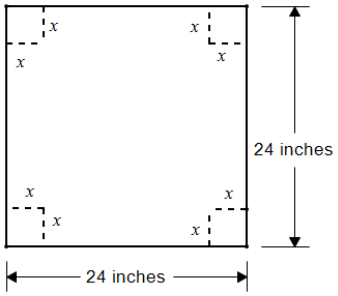
Solve $\dfrac{x-5}{\left|x+1\right|}\le 0$.

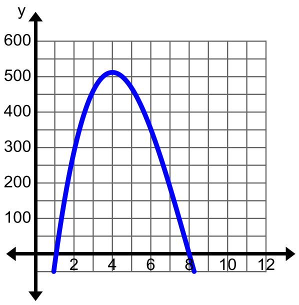
<table>
<tr><td>$\dfrac{x-5}{\left|x+1\right|}\le 0$</td><td>The expression on the left is undefined when $x=-1$. The expression is zero when $x=5$.</td></tr>
<tr><td> </td><td>Create a sign chart, using where the function is zero or undefined, to determine where the expression is negative or equal to zero.</td></tr>
<tr><td>The solution is $(-\infty,-1)\cup(-1,5]$.</td><td>
The graph confirms the solution.</td></tr>
</table>

Example 10:
The length of a rectangle is five more than the twice the width. If the area is at least 75 square centimeters, what are the possible values for the width?

$A = lw$ $l = 2w + 5$ $w(2w+5) \geq 75$	Write an inequality statement with the known information.
$w(2w+5) \geq 75$ $2w^2 + 5w - 75 \geq 0$ $(2w+15)(w-5) \geq 0$	Gather all the terms on one side of the inequality.
	Create a sign chart.
The width must be greater than or equal to 5 cm or $[5, \infty)$.	

Example 11:
A packaging company is designing a new open-topped box with a volume of at least 512 in^3. The box is to be made from a piece of cardboard measuring 24 inches by 24 inches by cutting identical squares from the corners and turning up the sides. Describe the possible lengths of the sides of the removed squares.



$512 \geq x(24-2x)(24-2x)$	The volume of a rectangular prism is $V = lwh$. The height is x. The width and length are $24-2x$. The length of the side of the square being cut out must be $0 < x < 12$.
$512 \geq x(24-2x)(24-2x)$ $512 \geq 4x^3 - 96x^2 + 576x$ $0 \geq 4x^3 - 96x^2 + 576x - 512$	Expand the equation and get all the terms on the same side so that the expression is compared to zero.
	Use technology to graph the function and find the zeros.
The volume of the box will equal or exceed 512 in^3 if the removed square has a side length on the interval $[1.072, 8]$.	

Practice Exercises C

1. $x^2 + x - 12 \geq 0$

2. $x^2 + 11x + 28 < 0$

3. $x^2 + 3x \geq 4$

4. $4x^3 - 4x > 0$

5. $x^3 + 2x^2 - 15x < 0$

6. $(x+1)(x^2 - 3x + 2) < 0$

7. $x^3 - 6x^2 \leq 7x$

8. $(x+1)(x-3)^2 > 0$

9. $\dfrac{x}{x+3} \geq 0$

10. $\dfrac{x-1}{x^2-4} < 0$

11. $\dfrac{x+2}{x^2-9} \leq 0$

12. $\dfrac{x^2-4}{x^2+4} \geq 0$

13. $x|x-2| > 0$

14. $(2x-1)\sqrt{x+4} < 0$

15. $(3x-4)\sqrt{2x+1} \geq 0$

16. The perimeter of a rectangle is 60 feet. Describe the possible lengths of a side if the area of the rectangle is not to exceed 161 square feet.

17. A diver leaps into the air at 20 feet per second from a diving board that is 12 feet above the water. For how many seconds is the diver at least 10 feet above the water?

18. A projectile is fired straight upward from ground level with an initial velocity of 96 feet per second. During which interval of time will the projectile's height exceed 80 feet?

19. An open box is made from a rectangular piece of cardboard measuring 11 inches by 14 inches by cutting identical squares from the corners and turning up the sides. Describe the possible lengths of the sides of the removed squares if the volume of the open box is not to exceed 132 cubic inches.

20. A calculator company's fixed monthly cost is $25,000 and the cost of producing a single calculator is $75. Describe the company's production level for the month so that the average cost of producing a calculator does not exceed $125.

21. A new drink company is packaging their new cola in 1-liter (1000 cm^3) cylindrical cans. Find the radius of the cans if the cans have a surface area that is less than 500 cm^2.

Solving for a Specified Variable

Example 12: Solve $\frac{1}{c}-\frac{c}{a^2-b^2}=0$ for *c*.

$\frac{1}{c}-\frac{c}{a^2-b^2}=0$	
$\frac{1}{c}-\frac{c}{a^2-b^2}+\frac{c}{a^2-b^2}=0+\frac{c}{a^2-b^2}$ $\frac{1}{c}=\frac{c}{a^2-b^2}$	Add $\frac{c}{a^2-b^2}$ to both sides of the equation.
$1\left(a^2-b^2\right)=c\cdot c$ $a^2-b^2=c^2$	Cross multiply.
$\left(a^2-b^2\right)^{1/2}=\left(c^2\right)^{1/2}$ $\pm\sqrt{a^2-b^2}=c$	Use the properties of exponents to eliminate the square and simplify.

Example 13: Solve $\mu=\sqrt{\frac{3RT}{M}}$ for *R*.

$\mu=\sqrt{\frac{3RT}{M}}$	
$\mu^2=\left(\sqrt{\frac{3RT}{M}}\right)^2$ $\mu^2=\frac{3RT}{M}$	Use the properties of exponents to eliminate the square root.
$M\mu^2=3RT$	Multiply both sides of the equation by *M*.
$\frac{M\mu^2}{3T}=R$	Divide by 3*T*.

Practice Exercises D

Solve each equation for the specified variable.

1. $\frac{q}{m}=\frac{2V}{B^2r^2}$ solve for B

2. $\frac{l}{T^2}=\frac{g}{4\pi^2}$ solve for T

3. $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ solve for c

4. $\frac{r_1}{r_2}=\sqrt{\frac{M_2}{M_1}}$ solve for M_1

5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ solve for y

6. $T=\frac{24\left(R-r\right)}{L}$ solve for R

7. $\sqrt{b^2-4ac}=k$ solve for b

8. $4p\left(y-k\right)=\left(x-h\right)^2$ solve for x

9. $S=\frac{n}{2}\left(a_1+a_n\right)$ solve for a_n

10. $a_n=a_1+\left(n-1\right)d$ solve for n.

11. $V=\frac{4}{3}\pi r^3$ solve for r

12. $y=\sqrt{a^2-\frac{a^2x^2}{b^2}}$ solve for b.

13. $\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}$ solve for r_2

14. $\frac{V^2}{R^2}=\frac{2g}{R+h}$ solve for h

15. $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$ solve for y_2

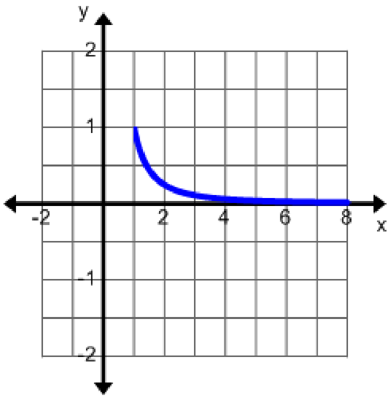
16. $A=\frac{2Tt+Qq}{2T+Q}$ for Q

Unit 4 Cluster 1 (A.CED.2 and A.CED.3): Writing and Solving Equations and Inequalities in Two Variables

Cluster 1: Create equations that describe numbers or relationships
- 4.1 Create equations in two or more variables to represent relationships between quantities
- 4.1 Graph equations on coordinate axes with labels and scales
- 4.1 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context

Example 1:

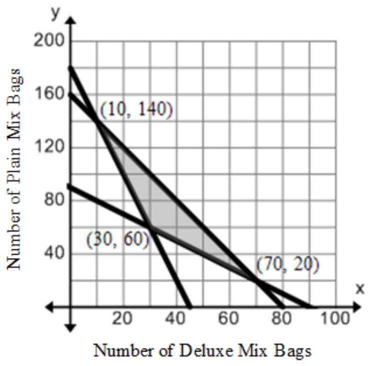
Given the sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...$ write and graph the rational equation that models the relationship between the term in the sequence and its value.

<table>
<tr><td>$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...$</td><td></td></tr>
<tr><td>
<table>
<tr><th>Term</th><th>Value</th></tr>
<tr><td>1</td><td>$1 = \frac{1}{1}$</td></tr>
<tr><td>2</td><td>$\frac{1}{2} = \frac{1}{2^2}$</td></tr>
<tr><td>3</td><td>$\frac{1}{9} = \frac{1}{3^2}$</td></tr>
<tr><td>4</td><td>$\frac{1}{16} = \frac{1}{4^2}$</td></tr>
</table>
</td><td>The general term is $\frac{1}{n^2}$.</td></tr>
<tr><td>$f(x) = \frac{1}{x^2}$ when $x \geq 1$ and an integer.
 </td><td>Note: this is a graph of the rational equation, not the graph of the sequence.</td></tr>
</table>

Example 2:
All-a-Shirt budgets $6000 to restock 200 shirts. T-shirts sell for $12, polo for $24, and rugby for $36. You need to buy twice as many rugby shirts as polo shirts. If you buy all three types of shirts, how many of each type should you buy?

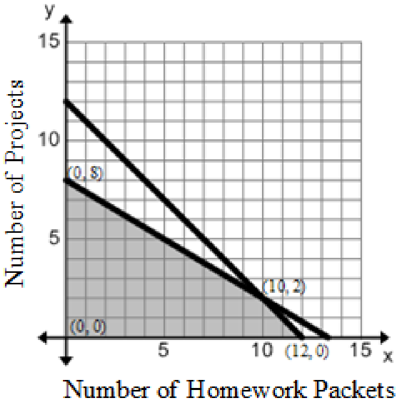
Relationship of polo to rugby: $z=2y$ Total number of shirts: $x+y+z=200$ Total cost: $12x+24y+36z=6000$	Write the equations. Let *x* represent the number of t-shirts, *y* represent the number of polo shirts, and *z* represent the number of rugby shirts.
$x+y+2y=200$ $x+3y=200$ $12x+24y+36\left(2y\right)=6000$ $12x+24y+72y=6000$ $12x+96y=6000$	Use the relationship of the polo to rugby shirts to rewrite the equations in terms of two variables.
$x+3y=200$ $12x+96y=6000$ $x=20,\ y=60,\ z=120$	Solve the system of equations using the method of your choice.
You should order 20 t-shirts, 60 polo shirts, and 120 rugby shirts.	

Example 3:
The Sweet Tooth Candy Shoppe is purchasing a candy mix with two types of chocolate: dark chocolate and white chocolate. They need at least 180 pounds of dark chocolate and 90 pounds of white. Their supplier has two mixes for them to buy. The deluxe mix costs $10.00 a bag and has 4 pounds of dark and 1 pound of white. The plain mix costs $5.00 a pound of each. The Sweet Tooth Candy Shoppe can pay at most $800 for the chocolate. How many bags of each can be purchased? Use a graph to help you decide.

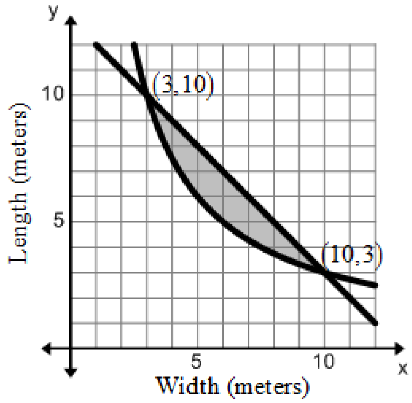
$4x+y\ge180$ $\quad x\ge0$ $x+y\ge90$ $\quad y\ge0$ $10x+5y\le800$	Let *x* represent the number of deluxe bags and *y* represent the number of plain bags. Write all of the constraint equations.
	Graph to find the solution area.

$10(10)+5(140)\le 800$ $800\le 800$ $10(70)+5(20)\le 800$ $800\le 800$ $10(30)+5(60)\le 800$ $600\le 800$	Any point in the shaded area is a solution. However, to minimize the cost check all three of the intersection points in $10x+5y$ to see which one is less than \$800.
Buying 30 bags of the deluxe mixture and 60 bags of the plain mixture will give you the required pounds of chocolate for the least amount of money.	

Example 4:
For his math grade, Carter can do extra homework packets for 70 points each or math projects for 80 points each. He estimates that each homework packet will take 9 hours and each project will take 15 hours and that he will have at most 120 hours to spend. He may turn in a total of no more than 12 packets or projects. How many of each should he complete in order to receive the highest score?

$x+y\le 12$ $x\ge 0$ $9x+15y\le 120$ $y\ge 0$ $70x+80y=\text{maximum}$	Let *x* represent the number of homework packets and *y* represent the number of projects. Write all of the constraint equations.
	Graph to find the solution area.
$70(0)+80(0)=0$ $70(12)+80(0)=840$ $70(10)+80(2)=860$ $70(0)+80(8)=640$	Any point in the shaded region is a solution. To find the highest score, substitute the intersection points into the expression $70x+80y$ to see which one is the greatest.
Completing 10 homework packets and 2 projects will maximize Carter’s grade.	

Example 5:
The perimeter of a rectangle is at most 26 meters. Its area is at least 30 square meters. What are the possible dimensions of the rectangle?

$2(x+y)\le 26$ $xy\ge 30$ $x\ge 0$ $y\ge 0$	Write the constraints.
	Graph to find the solution area.
Any point in the shaded region is a solution. For example, $(5,7)$ is a solution because the perimeter would be $2(5+7)=24$ meters which is less than 26 meters. The area would be $5(7)=35$ square meters which is greater than 30 square meters.	

Practice Exercises A

For the sequence, write and graph the rational equation that models the relationship between the term in the sequence and its value.

1. $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},...$

2. $1,\frac{1}{8},\frac{1}{27},\frac{1}{64},\frac{1}{125},...$

3. $\frac{1}{3},\frac{1}{6},\frac{1}{9},\frac{1}{12},...$

4. $2,\frac{3}{4},\frac{4}{9},\frac{5}{16},...$

5. $-\frac{1}{2},0,\frac{1}{4},\frac{2}{5},\frac{1}{2},\frac{4}{7},...$

6. $1,\frac{4}{3},\frac{3}{2},\frac{8}{5},\frac{5}{3},\frac{12}{7},...$

Write the constraints for each situation and graph the solution area.

7. A manufacturer produces the following two items: backpacks and messenger bags. Each item requires processing in each of two departments. The cutting department has 60 hours available and finishing department has 42 hours available each week for production. To manufacture a backpack requires 4 hours in cutting and 3 hours in finishing while a messenger bag requires 3 hours in cutting and 2 hours in finishing. Profits on the items are $10 and $7 respectively. If all the bags can be sold, how many of each should be made to maximize profits?

8. Olivia's Orchard consists of 240 acres upon which she wishes to plant red delicious and honey crisp apples. Profit per acre of red delicious is $400. Profit per acre for honey crisp is $300. Furthermore, the total number of hours of labor available during harvest is 3200. Each acre of red delicious requires 20 hours of labor. Each acre of honey crisp requires 10 hours of labor. Determine how the land should be divided to maximize the profits.

9. Kathy owns a car and a moped. She has at most 12 gallons of gasoline to be used between the car and the moped. The car's tank holds at most 18 gallons and the moped's 3 gallons. The mileage for the car is 20 mpg. The mileage for the moped is 100 mpg. How many gallons of gasoline should each vehicle use if Kathy wants to travel as far as possible? What is the maximum number of miles?

10. Cohen is about to take a test that contains short answer questions worth 4 points each and word problems worth 7 points each. Cohen must do at least 5 short answer questions but time restricts doing more than 10. He must do at least 3 word problems but time restricts doing more than 10. Cohen can do no more than 18 questions in total. How many of each type of question must Cohen do in order to maximize his score?

11. Bob's Furniture produces chairs and sofas. The chairs require 20 feet of wood, 1 pound of foam rubber, and 2 square yards of fabric. The sofas require 100 feet of wood, 50 pounds of foam rubber, and 20 square yards of fabric. The company has 1900 feet of wood, 500 pounds of foam rubber, and 240 square yards of fabric. The chairs can be sold for $80 and the sofas for $1,200. How many of each should be produced to maximize the income?

12. The perimeter of the base of a box is no more than 60 inches. If the height is fixed at 8 inches and the volume is at least 1000 cubic inches, what are three possible dimensions of the base of the box?

13. Jen earns $10 per hour for tutoring and $7 per hour as a teacher's aide. Jen must have enough time for studies so she can work no more than 20 hours per week. She must spend at least 3 hours per week tutoring and no more than 8 hours per week tutoring. How many hours a week will she spend tutoring and working as a teacher's aide to maximize the amount she earns?

14. Piper’s Paper producers has two factories that produce three types of paper: low, medium, and high grade paper. It supplies no more than 24 tons of low grade, 6 tons of medium grade, and 30 tons of high grade paper. The east factory produces 8 tons of low grade, 1 ton of medium grade, and 2 tons of high grade paper daily and costs $2000 per day to operate. The west factory produces 2 tons of low grade, 1 ton of medium grade, and 8 tons of high grade paper daily and costs $4000 per day to operate. How many days should each factory operate to fill the orders at minimum cost if each factory must produce part of the product?

15. Sally’s Scrapbooking prints pages of photographs for albums. A page containing 4 photos will cost $3 while a page containing 6 photos will cost $5. Cyndi can spend no more than $90 for photo pages of her recent vacation and can use no more than 20 pages in her album. What combination of 4-photo pages and 6-photo pages will maximize the number of photos she can display? How many photos can she display?

Unit 4 Cluster 4 (F.BF.1 and F.BF.1c): Combining Functions

Cluster 4: Build a function that models a relationship between two quantities.
4.3 Combine standard function types using arithmetic operations.

Honors
4.3H Compose functions.

Combining functions using arithmetic operations

Let *f* and *g* be any two functions. A new function *h* can be created by performing any of the four basic operations on *f* and *g*.

Operation	**Definition**	**Example:** $f(x)=5x^2+2x$, $g(x)=-3x^2$
Addition	$h(x)=(f+g)(x)$	$h(x)=5x^2+2x+(-3x^2)=2x^2+2x$
Subtraction	$h(x)=(f-g)(x)$	$h(x)=5x^2+2x-(-3x^2)=8x^2+2x$
Multiplication	$h(x)=(fg)(x)$	$h(x)=(5x^2+2x)\cdot(-3x^2)=-15x^4-6x^3$
Division	$h(x)=\left(\frac{f}{g}\right)(x)$	$h(x)=\frac{5x^2+2x}{-3x^2}=\frac{\cancel{x}(5x+2)}{\cancel{x}(-3x)}=\frac{5x+2}{-3x}$

The domain of *h* consists of the *x*-values that are in the domains of both *f* and *g*. Additionally, the domain of a quotient does not include *x*-values for which $g(x)=0$.

Example 1:

Let $f(x)=x^2+1$ and $g(x)=\sqrt{x+3}-2$. Find an algebraic expression for $h(x)$ and determine its domain if:

a. $h(x)=(f+g)(x)$

b. $h(x)=(f-g)(x)$

c. $h(x)=(fg)(x)$

d. $h(x)=\left(\frac{f}{g}\right)(x)$

$f(x)=x^2+1$ domain is $(-\infty,\infty)$. $g(x)=\sqrt{x+3}-2$ domain is $[-3,\infty)$. The intersection is $[-3,\infty)$.	Determine the domains of $f(x)$ and $g(x)$. Find the intersection of their domains. The intersection will be the domain for the new functions obtained through arithmetic operations. When dividing functions there may be an additional restriction.

a. $h(x)=(f+g)(x)$ $h(x)=\left(x^2-1\right)+\left(\sqrt{x+3}-2\right)$ $h(x)=x^2+\sqrt{x+3}-3$	Find the sum of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=x^2+\sqrt{x+3}-3$ and its domain is $[-3,\infty)$.	

b. $h(x)=(f-g)(x)$ $h(x)=\left(x^2-1\right)-\left(\sqrt{x+3}-2\right)$ $h(x)=x^2-\sqrt{x+3}+1$	Find the difference of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=x^2-\sqrt{x+3}+1$ and its domain is $[-3,\infty)$.	

c. $h(x)=(fg)(x)$ $h(x)=\left(x^2-1\right)\left(\sqrt{x+3}-2\right)$ $h(x)=x^2\sqrt{x+3}-2x^2-\sqrt{x+3}+2$	Find the product of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=x^2\sqrt{x+3}-2x^2-\sqrt{x+3}+2$ and its domain is $[-3,\infty)$.	

d. $h(x)=\left(\frac{f}{g}\right)(x)$ $h(x)=\frac{x^2-1}{\sqrt{x+3}-2}$	Find the quotient of the algebraic expressions representing $f(x)$ and $g(x)$. Notice that $x\neq 3$ or the denominator will be zero.
The new function is $h(x)=\frac{x^2-1}{\sqrt{x+3}-2}$ and its domain is $(-3,\infty)$.	

Example 2:

Let $f(x)=x^3$ and $g(x)=\frac{x+1}{x+3}$. Find an algebraic expression for $h(x)$ and determine its domain if:

a. $h(x)=(f+g)(x)$

b. $h(x)=(g-f)(x)$

c. $h(x)=(fg)(x)$

d. $h(x)=\left(\frac{g}{f}\right)(x)$

<table>
<tr><td>$f\left(x\right)=x^3$ domain is $\left(-\infty,\infty\right)$.
$g\left(x\right)=\dfrac{x+1}{x+3}$ domain is $\left(-\infty,-3\right)\cup\left(-3,\infty\right)$.
The intersection is $\left(-\infty,-3\right)\cup\left(-3,\infty\right)$.</td><td>Determine the domains of $f\left(x\right)$ and $g\left(x\right)$. Find the intersection of their domains.
The intersection will be the domain for the new functions obtained through arithmetic operations. When dividing functions there may be an additional restriction.</td></tr>
</table>

<table>
<tr><td>a. $h\left(x\right)=\left(f+g\right)\left(x\right)$
$h\left(x\right)=\left(x^3\right)+\left(\dfrac{x+1}{x+3}\right)$
$h\left(x\right)=\dfrac{x^3\left(x+3\right)+x+1}{x+3}$
$h\left(x\right)=\dfrac{x^4+3x^3+x+1}{x+3}$</td><td>Find the sum of the algebraic expressions representing $f\left(x\right)$ and $g\left(x\right)$.</td></tr>
<tr><td colspan="2">The new function is $h\left(x\right)=\dfrac{x^4+3x^3+x+1}{x+3}$ and its domain is $\left(-\infty,-3\right)\cup\left(-3,\infty\right)$.</td></tr>
</table>

<table>
<tr><td>b. $h\left(x\right)=\left(g-f\right)\left(x\right)$
$h\left(x\right)=\left(\dfrac{x+1}{x+3}\right)-\left(x^3\right)$
$h\left(x\right)=\dfrac{x+1-x^3\left(x+3\right)}{x+3}$
$h\left(x\right)=\dfrac{x+1-x^4-3x^3}{x+3}$</td><td>Find the difference of the algebraic expressions representing $f\left(x\right)$ and $g\left(x\right)$.</td></tr>
<tr><td colspan="2">The new function is $h\left(x\right)=\dfrac{-x^4-3x^3+x+1}{x+3}$ and its domain is $\left(-\infty,-3\right)\cup\left(-3,\infty\right)$.</td></tr>
</table>

<table>
<tr><td>c. $h\left(x\right)=\left(fg\right)\left(x\right)$
$h\left(x\right)=\left(x^3\right)\left(\dfrac{x+1}{x+3}\right)$
$h\left(x\right)=\dfrac{x^4+x^3}{x+3}$</td><td>Find the product of the algebraic expressions representing $f\left(x\right)$ and $g\left(x\right)$.</td></tr>
<tr><td colspan="2">The new function is $\dfrac{x^4+x^3}{x+3}$ and its domain is $\left(-\infty,-3\right)\cup\left(-3,\infty\right)$.</td></tr>
</table>

d. $h(x)=\left(\frac{g}{f}\right)(x)$ $h(x)=\frac{\frac{x+1}{x+3}}{x^3}$ $h(x)=\frac{x+1}{x^3(x+3)}$	Find the quotient of the algebraic expressions representing $f(x)$ and $g(x)$. Notice that $x \neq 0$ or the denominator will be zero.
The new function is $h(x)=\frac{x+1}{x^3(x+3)}$ and its domain is $(-\infty,-3)\cup(-3,0)\cup(0,\infty)$.	

Example 3:

Let $f(x)=\cos x$ and $g(x)=2^x+1$. Find an algebraic expression for $h(x)$ and determine its domain if:

a. $h(x)=(f+g)(x)$

b. $h(x)=(f-g)(x)$

c. $h(x)=(fg)(x)$

d. $h(x)=\left(\frac{f}{g}\right)(x)$

$f(x)=\cos x$ domain is $(-\infty,\infty)$. $g(x)=2^x+1$ domain is $(-\infty,\infty)$. The intersection is $(-\infty,\infty)$.	Determine the domains of $f(x)$ and $g(x)$. Find the intersection of their domains. The intersection will be the domain for the new functions obtained through arithmetic operations. When dividing functions there may be an additional restriction.

a. $h(x)=(f+g)(x)$ $h(x)=(\cos x)+\left(2^x+1\right)$ $h(x)=\cos x+2^x+1$	Find the sum of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=\cos x+2^x+1$ and its domain is $(-\infty,\infty)$.	

b. $h(x)=(f-g)(x)$ $h(x)=(\cos x)-(2^x+1)$ $h(x)=\cos x-2^x-1$	Find the difference of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=\cos x-2^x-1$ and its domain is $(-\infty,\infty)$.	

c. $h(x)=(fg)(x)$ $h(x)=(\cos x)(2^x+1)$ $h(x)=2^x\cos x+\cos x$	Find the product of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=2^x\cos x+\cos x$ and its domain is $(-\infty,\infty)$.	

d. $h(x)=\left(\frac{f}{g}\right)(x)$ $h(x)=\frac{\cos x}{2^x+1}$	Find the quotient of the algebraic expressions representing $f(x)$ and $g(x)$. Notice that the denominator will never be zero so there is no restriction.
The new function is $h(x)=\frac{\cos x}{2^x+1}$ and its domain is $(-\infty,\infty)$.	

Practice Exercises A

Find an algebraic expression for $h(x)$ and determine its domain .

1. $f(x)=\sqrt{x-4}+2$ and $g(x)=-3x^2$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(f-g)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\frac{f}{g}\right)(x)$

2. $f(x)=4^{x-2}+1$ and $g(x)=\sqrt{2x}$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(g-f)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\frac{g}{f}\right)(x)$

3. $f(x)=\sin x$ and $g(x)=x^3-3$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(f-g)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\frac{f}{g}\right)(x)$

4. $f(x)=x^2-5x-6$ and $g(x)=-3x+1$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(g-f)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\frac{g}{f}\right)(x)$

5. $f(x)=x^3-x^2$ and $g(x)=x^2-7x+6$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(f-g)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\frac{f}{g}\right)(x)$

6. $f(x)=\frac{x-3}{x+5}$ and $g(x)=x+3$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(g-f)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\frac{g}{f}\right)(x)$

7. $f(x)=\cos 3x$ and $g(x)=\sqrt[3]{x+1}$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(f-g)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\frac{f}{g}\right)(x)$

8. $f(x)=\sqrt{x+7}$ and $g(x)=\frac{1}{x}$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(g-f)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\frac{g}{f}\right)(x)$

Evaluating Combined Functions

Example 4:

Let $f(x)=\sqrt{x+2}-2$ and $g(x)=\dfrac{x-4}{x+3}$. Evaluate each of the following:

a. $2f(7)+g(3)$

b. $g(4)-f(-2)$

c. $f(14)\cdot 2g(-1)$

d. $\dfrac{-f(3)}{g(1)}$

a. $2f(7)+g(3)$ $2\left(\sqrt{7+2}-2\right)+\dfrac{3-4}{3+3}$ $2\left(\sqrt{9}-2\right)+\dfrac{-1}{6}$ $2(3-2)-\dfrac{1}{6}$ $2-\dfrac{1}{6}$ $\dfrac{11}{6}$	Substitute $x=7$ in to $f(x)$ and $x=3$ into $g(x)$ and simplify.

b. $g(4)-f(-2)$ $\dfrac{4-4}{4+3}-\left(\sqrt{-2+2}-2\right)$ $\dfrac{0}{7}-\left(\sqrt{0}-2\right)$ $0-(-2)$ $0+2$ 2	Substitute $x=-2$ in to $f(x)$ and $x=4$ into $g(x)$ and simplify.

c. $f(14)\cdot 2g(-1)$ $\left(\sqrt{14+2}-2\right)(2)\left(\dfrac{-1-4}{-1+3}\right)$ $\left(\sqrt{16}-2\right)(2)\left(\dfrac{-5}{2}\right)$ $(4-2)(-5)$ $(2)(-5)$ -10	Substitute $x=-2$ in to $f(x)$ and $x=4$ into $g(x)$ and simplify.

Practice Exercises B

Let $a(x)=x^2-2$, $b(x)=\sqrt{x+1}$, $c(x)=5^{x-3}-2$, $d(x)=2\cos x$, and $f(x)=\frac{x}{x-4}$. Evaluate each of the following.

1. $d(\pi)+3f(2)$	2. $-2c(3)+f(1)$	3. $a(-2)+b(3)$
4. $4d(2\pi)-b(8)$	5. $a(-3)-c(4)$	6. $a(0)-4f(0)$
7. $c(2)\cdot b(-1)$	8. $d\left(\frac{\pi}{3}\right)\cdot a(-3)$	9. $f(5)\cdot c(4)$
10. $\frac{a(-2)}{3b(0)}$	11. $\frac{f(3)}{a(1)}$	12. $\frac{c(1)}{d(-\pi)}$

13. A company estimates that its cost and revenue can be modeled by the functions $C(x)=-0.75x^2+100x+20{,}000$ and $R(x)=150x+100$ where *x* is the number of units produced. The company's profit, *P*, is modeled by $R(x)-C(x)$. Find the profit equation and determine the profit when 1,000,000 units are produced.

14. Consider the demand equation $p(x)=-\frac{1}{15}x+30;\ 0\le x\le 450$ where *p* represents the price and *x* the number of units sold. Write an equation for the revenue, *R*, if the revenue is the price times the number of units sold. What is the revenue if 225 units are sold?

15. The average Cost $\overline{C}$ of manufacturing *x* computers per day is obtained by dividing the cost function by the number of computers produced that day, *x*. If the cost function is $C(x)=0.5x^3-34x^2+1213x$, find an equation for the average cost of manufacturing. What is the average cost of producing 100 computers per day?

16. The service committee wants to organize a fund-raising dinner. The cost of renting a facility is \$300 plus \$5 per chair or $C(x)=5x+300$, where *x* represents the number of people attending the fund-raiser. The committee wants to charge attendees \$30 each or $R(x)=30x$. How many people need to attend the fund-raiser for the event to raise \$1,000?

Composition of Functions (HONORS)

VOCABULARY

Composing one function with another function is applying one function to the result of another function. The notation for composition is $(f \circ g)(x)$ or $f(g(x))$ and $(g \circ f)(x)$ or $g(f(x))$. The inner function is always evaluated in the outer function.

The domain of the composite function is determined by the domain of the inside function and the composite function.

Example 5:

Given $f(x) = x^2 + 2x + 1$ and $g(x) = x + 5$, find $(f \circ g)(x)$ and its domain.

$g(x) = x + 5$ There are no exclusions on the function.	$g(x)$ will be the inside function, therefore, determine the values that need to be excluded from its domain.
$(f \circ g)(x) = (x+5)^2 + 2(x+5) + 1$ $(f \circ g)(x) = (x^2 + 10x + 25) + (2x + 10) + 1$ $(f \circ g)(x) = x^2 + 12x + 36$	Find the composite. Substitute $g(x)$ into every x in $f(x)$ and simplify.
$(f \circ g)(x) = x^2 + 12x + 36$ There are no exclusions on the function.	Find the domain of $(f \circ g)(x)$.
The composite function is $(f \circ g)(x) = x^2 + 12x + 36$ and its domain is $(-\infty, \infty)$.	

Example 6:

Given $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{x}{x-2}$, find $(f \circ g)(x)$ and its domain.

$g(x) = \frac{x}{x-2}$ $x - 2 = 0$ $x = 2$	$g(x)$ will be the inside function, therefore, determine the values that need to be excluded from its domain.

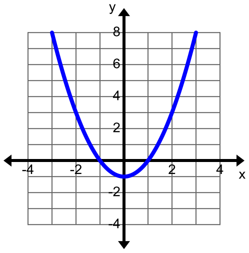
$(f \circ g)(x) = \dfrac{1}{\frac{x}{x-2}+1}$ $(f \circ g)(x) = \dfrac{1}{\frac{x+(x-2)}{x-2}}$ $(f \circ g)(x) = \dfrac{1}{\frac{2x-2}{x-2}}$ $(f \circ g)(x) = \dfrac{x-2}{2x-2}$	Find the composite. Substitute $g(x)$ into every x in $f(x)$ and simplify.
$(f \circ g)(x) = \dfrac{x-2}{2x-2}$ $2x-2=0$ $2x=2$ $x=1$	Find the domain of $(f \circ g)(x)$.
The composite function is $(f \circ g)(x) = \dfrac{x-2}{2x-2}$ and its domain is $(-\infty,1)\cup(1,2)\cup(2,\infty)$.	

Example 7:
Given $f(x) = x-3$ and $g(x) = 2^x$, find $(g \circ f)(x)$ and its domain.

$f(x) = x-3$ There are no exclusions on the function.	$f(x)$ will be the inside function, therefore, determine the values that need to be excluded from its domain.
$(g \circ f)(x) = 2^{x-3}$	Find the composite. Substitute $f(x)$ into every x in $g(x)$ and simplify.
$(g \circ f)(x) = 2^{x-3}$ There are no exclusions on the function.	Find the domain of $(g \circ f)(x)$.
The composite function is $(g \circ f)(x) = 2^{x-3}$ and its domain is $(-\infty,\infty)$.	

Example 8:

Given $f(x)=x^2-1$ and $g(x)=\sqrt{x}$, find $(g\circ f)(x)$ and its domain.

$f(x)=x^2-1$ There are no exclusions on the function.	$f(x)$ will be the inside function, therefore, determine the values that need to be excluded from its domain.
$(g\circ f)(x)=\sqrt{x^2-1}$	Find the composite. Substitute $f(x)$ into every x in $g(x)$ and simplify.
$(g\circ f)(x)=\sqrt{x^2-1}$ $x^2-1\geq 0$ $(x-1)(x+1)\geq 0$  The function is positive on the interval $(-\infty,-1)\cup(1,\infty)$.	Find the domain of $(g\circ f)(x)$.
The composite function is $(g\circ f)(x)=\sqrt{x^2-1}$ and its domain is $(-\infty,-1)\cup(1,\infty)$.	

Practice Exercises C

Find the indicated composite function and its domain.

1. $f(x) = x - 8$ and $g(x) = \dfrac{1}{x-7}$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

2. $f(x) = 5^{x-4}$ and $g(x) = x^2 - 4$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

3. $f(x) = \sqrt{x-6}$ and $g(x) = x^2 - 3$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

4. $f(x) = |x-5| - 2$ and $g(x) = -2\sin x$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

5. $f(x) = \sqrt[3]{x-2}$ and $g(x) = x^3 + 2$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

6. $f(x) = \dfrac{1}{2x}$ and $g(x) = e^x$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

7. $f(x) = \cos x$ and $g(x) = 4 - x$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

8. $f(x) = \sqrt[3]{x+5}$ and $g(x) = -2|x-4|$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

Evaluating Composite Functions

Example 9:

Let $f(x)=(x-1)^2+3$ and $g(x)=-2|x+4|+5$. Evaluate each of the following:

a. $(f\circ g)(-2)$

b. $(g\circ f)(0)$

c. $(f\circ f)(1)$

d. $(g\circ g)(-7)$

a. $(f\circ g)(-2)=f(g(-2))$ $g(-2)=-2\|-2+4\|+5$ $g(-2)=-2\|2\|+5$ $g(-2)=-4+5$ $g(-2)=1$	Evaluate the inside function $g(x)$ at $x=-2$.
$f(1)=(1-1)^2+3$ $f(1)=0^2+3$ $f(1)=3$	Evaluate the outside function $f(x)$ at the value of $g(-2)$ or $x=1$.
$(f\circ g)(-2)=3$	

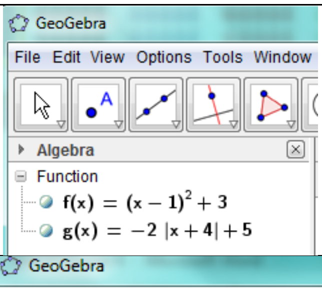
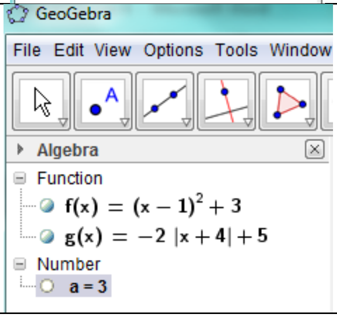
b. $(g\circ f)(0)=g(f(0))$ $f(0)=(0-1)^2+3$ $f(0)=(-1)^2+3$ $f(0)=1+3$ $f(0)=4$	Evaluate the inside function $f(x)$ at $x=0$.
$g(4)=-2\|4+4\|+5$ $g(4)=-2\|8\|+5$ $g(4)=-16+5$ $g(5)=-11$	Evaluate the outside function $g(x)$ at the value of $f(0)$ or $x=4$.
$(g\circ f)(0)=-11$	

c. $(f \circ f)(1) = f(f(1))$ $f(1) = (1-1)^2 + 3$ $f(1) = 0^2 + 3$ $f(1) = 3$	Evaluate the inside function $f(x)$ at $x = 1$.
$f(3) = (3-1)^2 + 3$ $f(3) = 2^2 + 3$ $f(3) = 7$	Evaluate the outside function $f(x)$ at the value of $f(1)$ or $x = 3$.
$(f \circ f)(1) = 7$	

d. $(g \circ g)(-7) = g(g(-7))$ $g(-7) = -2\|-7+4\| + 5$ $g(-7) = -2\|-3\| + 5$ $g(-7) = -6 + 5$ $g(-7) = -1$	Evaluate the inside function $g(x)$ at $x = -7$.
$g(-7) = -2\|-1+4\| + 5$ $g(-7) = -2\|3\| + 5$ $g(-7) = -6 + 5$ $g(-7) = -1$	Evaluate the outside function $g(x)$ at the value of $g(-7)$ or $x = -1$.
$(g \circ g)(-7) = -1$	

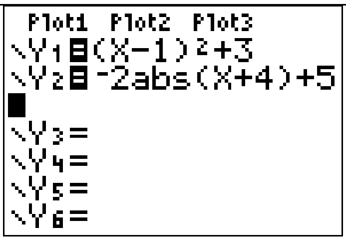
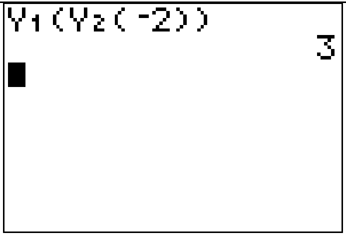
Evaluating Composite Functions using Geogebra

Let $f(x)=(x-1)^2+3$ and $g(x)=-2|x+4|+5$. Find $(f\circ g)(-2)$.

With the Algebra window and the Graphics window showing, enter the two functions in the input bar so that they are graphed. They will appear in the Algebra window.	 Function f(x) = (x − 1)² + 3 g(x) = −2 |x + 4| + 5
Type $f(g(-2))$ in the input bar then push enter and Geogebra will evaluate the composite function.	 Function f(x) = (x − 1)² + 3 g(x) = −2 |x + 4| + 5 Number a = 3

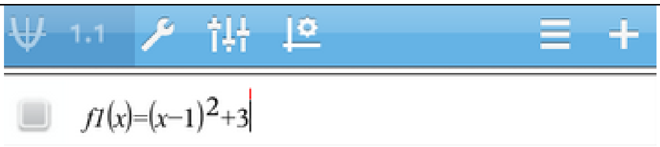
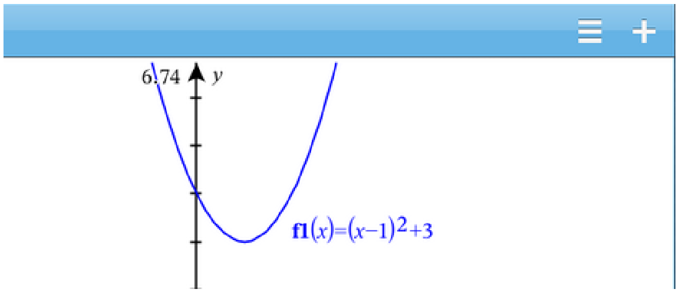
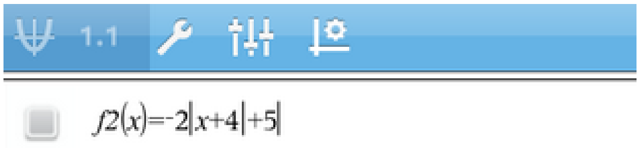
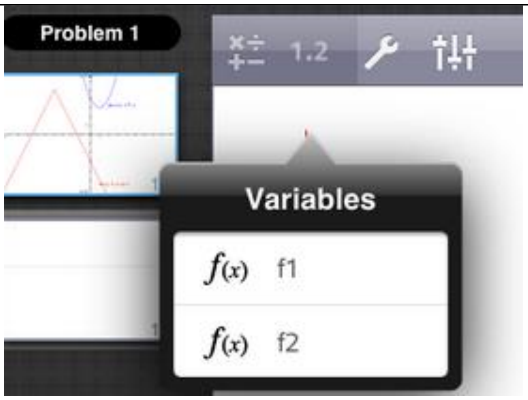
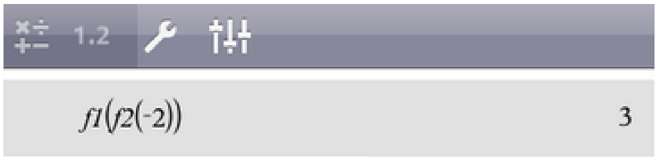
Evaluating Composite Functions using the TI-84 Graphing Calculator

Let $f(x)=(x-1)^2+3$ and $g(x)=-2|x+4|+5$. Find $(f\circ g)(-2)$.

Push STAT PLOT F1 Y= and enter $f(x)$ in Y1 and $g(x)$ in Y2. Go back to the home screen by pushing 2ND QUIT MODE.	 Plot1 Plot2 Plot3 \Y1■(X−1)²+3 \Y2■-2abs(X+4)+5 ■ \Y3= \Y4= \Y5= \Y6=
Push DISTR VARS, arrow over to Y-VARS and select 1:Function by pushing L1 1 Y. $f(x)$ is in Y1 so push L1 1 Y. Put a parenthesis { (K in. Push DISTR VARS, arrow over to Y-VARS and select 1:Function by pushing L1 1 Y. $g(x)$ is in Y2 so push L2 2 Z. Put a parenthesis { (K in. Enter the value that the composite function is being evaluated at, -2, then close both parentheses. Push ENTRY SOLVE ENTER and the calculator will evaluate the composite function.	 Y1(Y2(-2)) 3 ■

Evaluating Composite Functions using the TI-*N*spire CAS App

Let $f(x)=(x-1)^2+3$ and $g(x)=-2|x+4|+5$. Find $(f\circ g)(-2)$.

Select a new document by pushing the + symbol in the upper left corner. Select Graphs so that you graph both functions. Type $f(x)$ in $f1(x)$ and push ENTER to graph it. To enter another equation push the + symbol at the upper right hand corner. Put $g(x)$ in $f2(x)$ and push ENTER to graph it.	 $f1(x)=(x-1)^2+3$   $f2(x)=-2\|x+4\|+5$
Add a new window by pushing the + in the upper left had corner. This time select Calculator. Push var and it will show you the functions that are available. Select f1 for $f(x)$. Push var again this time select f2 for $g(x)$. Enter -2 so that it will evaluate the composite at -2. Push ENTER and the answer will appear.	  f1(f2(-2)) 3

Practice Exercises D

Let $f\left(x\right)=2x^2+1$, $g\left(x\right)=\sqrt{x-2}$, and $h\left(x\right)=\dfrac{x-3}{x-5}$. Evaluate each composite function.

1. $\left(f\circ g\right)\left(6\right)$
2. $\left(g\circ f\right)\left(2\right)$
3. $\left(f\circ h\right)\left(4\right)$
4. $\left(h\circ f\right)\left(0\right)$
5. $\left(g\circ h\right)\left(6\right)$
6. $\left(h\circ g\right)\left(11\right)$
7. $\left(f\circ f\right)\left(-1\right)$
8. $\left(g\circ g\right)\left(18\right)$
9. $\left(h\circ h\right)\left(-1\right)$
10. You have a coupon for 15% off a meal at your favorite restaurant. You also have a $10 gift card to the restaurant.

    a. Write a function representing the cost, *x*, of the meal with the coupon.

    b. Write a function representing the cost, *x*, of the meal with the gift card.

    c. Does it save you more money if the coupon is applied first or if the gift card is applied first? How much is the savings?

11. In Utah the general sales tax is 5.95%. If you have a coupon for 10% off your entire bill, is it better to have the sales tax applied before or after the 10% off coupon?

12. A balloon's radius can be modeled by the equation $r\left(t\right)=0.05t+4$, where *t* is the time in seconds and *r* is measured in centimeters. The volume of a sphere is $V\left(r\right)=\frac{4}{3}\pi r^3$. Write the formula for $\left(V\circ r\right)\left(t\right)$. Find the volume of the balloon at 30 seconds.

Unit 4 Cluster 5 (F.BF.4): Inverses

Cluster 5: Build new functions from existing functions
- 4.5 Find inverse functions

Honors
- 4.5 Verify by composition that one function is the inverse of another.
- 4.5 Read values of an inverse function from a graph or a table, given that the function has an inverse.
- 4.5 Produce an invertible function from a non-invertible function by restricting the domain.

VOCABULARY

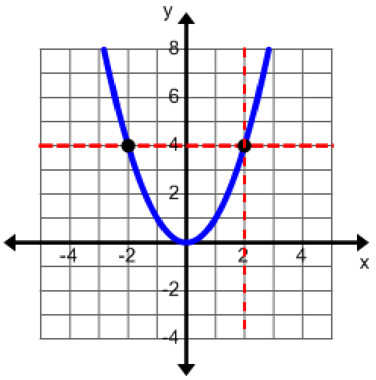
If no vertical line intersects the graph of a function *f* more than once, then *f* is a function. This is called the **vertical line test**. If no horizontal line intersects the graph of a function *f* more than once, then the inverse of *f* is itself a function. This is called the **horizontal line test**.

The **inverse of a function** is formed when the independent variable is exchanged with the dependent variable in a given relation. (Switch the *x* and *y* with each other.) A function takes a starting value, performs some operation on this value, and creates an output answer. The inverse of a function takes the output answer, performs some operation on it, and arrives back at the original function's starting value. Inverses are indicated by the notation f^{-1}.

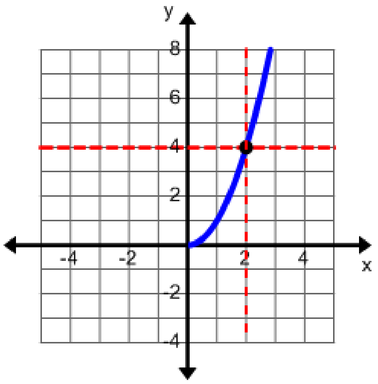
A function is a **one-to-one function** if and only if each second element corresponds to one and only one first element.

In order for the inverse of a function to be a function, the original function must be a one-to-one function and meets the criteria for the vertical and horizontal line tests.

Not all functions meet the criteria to have an inverse which is also a function. However, if the **domain is restricted**, or in other words only part of the domain is used, then the inverse will be a function.



This example is not one-to-one. It is a function because the vertical line intersects the graph only once. However, the horizontal line intersects the graph twice. There is an inverse to this example, but the inverse will not be a function.



This example is one-to-one. It is a function because the vertical and horizontal lines intersect the graph only once. The inverse will be a function.

Example 1:
Find the inverse of $f(x)=5x+7$.

$y=5x+7$	$f(x)$ is the same as *y* so replace $f(x)$ with *y*.
$x=5y+7$	Substitute *x* with *y* and *y* with *x*.
$x=5y+7$ $x-7=5y$ $\frac{x-7}{5}=y$	Solve for *y*.
$f^{-1}(x)=\frac{x-7}{5}$	This is the inverse of $f(x)$.

Example 2:
Find the inverse of $f(x)=x^2-6x+13$ when $x\le 3$.

$y=x^2-6x+13$	$f(x)$ is the same as *y* so replace $f(x)$ with *y*.
$x=y^2-6y+13$	Substitute *x* with *y* and *y* with *x*.
$x=\left(y^2-6y+9\right)+13-9$ $x=(y-3)^2+4$ $x-4=(y-3)^2$ $\pm\sqrt{x-4}=y-3$ $\pm\sqrt{x-4}+3=y$	Complete the square in order to solve for *y*.
$\pm\sqrt{x-4}+3=y$ $-\sqrt{x-4}+3=y$	The domain of the original function is the range of the inverse function. Therefore, we select the negative root.
$f^{-1}(x)=-\sqrt{x-4}+3$	This is the inverse of $f(x)$.

Example 3:

Find the inverse of $f\left(x\right)=\frac{4x+3}{3x-1}$.

$y=\frac{4x+3}{3x-1}$	$f\left(x\right)$ is the same as *y* so replace $f\left(x\right)$ with *y*.
$x=\frac{4y+3}{3y-1}$ $x\left(3y-1\right)=4y+3$ $3xy-x=4y+3$ $3xy-4y=x+3$ $y\left(3x-4\right)=x+3$ $y=\frac{x+3}{3x-4}$	Substitute each *x* with *y* and *y* with *x*.
$f^{-1}\left(x\right)=\frac{x+3}{3x-4}$	This is the inverse of $f\left(x\right)$.

Practice Exercises A

Find the inverse of each function.

1. $f\left(x\right)=-6x+8$
2. $f\left(x\right)=-\frac{1}{2}x-2$
3. $f\left(x\right)=3x-5$
4. $f\left(x\right)=-\frac{1}{4}x^2+3,\ x\le 0$
5. $f\left(x\right)=x^2-4x-12,\ x\ge 2$
6. $f\left(x\right)=2x^2+12x+14,\ x\ge -3$
7. $f\left(x\right)=\sqrt{x+4}$
8. $f\left(x\right)=\sqrt{2x-5}+4$
9. $f\left(x\right)=-2\sqrt{3-x}$
10. $f\left(x\right)=\sqrt{x+1}-3$
11. $f\left(x\right)=\frac{3x+5}{x-1}$
12. $f\left(x\right)=\frac{-2x+1}{5x-6}$
13. $f\left(x\right)=\frac{7x-6}{3x+2}$
14. $f\left(x\right)=\frac{4x-3}{x+4}$
15. $f\left(x\right)=\frac{1}{2}x^3-3$
16. $f\left(x\right)=-3x^3+7$
17. $f\left(x\right)=\left(x-2\right)^3+5$
18. $f\left(x\right)=\sqrt[3]{x+2}-3$
19. $f\left(x\right)=-2\sqrt[3]{x-5}+7$
20. $f\left(x\right)=\frac{1}{2}\sqrt[3]{4-x}+1$
21. $f\left(x\right)=\sqrt[3]{x-1}-4$

Composition of Inverse Functions

If *f* and f^{-1} are inverse functions, then $\left(f^{-1}\circ f\right)(x)=x$ and $\left(f\circ f^{-1}\right)(x)=x$ for all *x* in the domains of *f* and f^{-1} respectively.

Example 4:

Verify that $f(x)=3x-2$ and $g(x)=\dfrac{x+2}{3}$ are inverses of each other.

$(f\circ g)(x)=3\left(\dfrac{x+2}{3}\right)-2$ $(f\circ g)(x)=x+2-2$ $(f\circ g)(x)=x$	Find $(f\circ g)(x)$.
$(g\circ f)(x)=\dfrac{(3x-2)+2}{3}$ $(g\circ f)(x)=\dfrac{3x}{3}$ $(g\circ f)(x)=x$	Find $(g\circ f)(x)$.
Both $(f\circ g)(x)$ and $(g\circ f)(x)$ are equal to *x*, therefore, $f(x)$ and $g(x)$ are inverses of each other.	

Practice Exercises B

Verify that *f* and *g* are inverses are of each other.

1. $f(x)=\dfrac{x+3}{4}$ and $g(x)=4x-3$
2. $f(x)=x^3+1$ and $g(x)=\sqrt[3]{x-1}$
3. $f(x)=\dfrac{1}{2x}$ and $g(x)=\dfrac{1}{2x}$
4. $f(x)=\sqrt{x+2}-1$ and $g(x)=(x+1)^2-2,\ x\geq -1$
5. $f(x)=\dfrac{1}{2}x-4$ and $g(x)=2x+8$
6. $f(x)=(x-3)^3-2$ and $g(x)=\sqrt[3]{x+2}+3$
7. $f(x)=\dfrac{5}{x+1}$ and $g(x)=\dfrac{5-x}{x}$
8. $f(x)=\dfrac{1}{2}x^2-5,\ x\geq 0$ and $g(x)=\sqrt{2x+10}$
9. $f(x)=2x+4$ and $g(x)=\dfrac{1}{2}x-2$
10. $f(x)=\dfrac{x+3}{x-2}$ and $g(x)=\dfrac{2x+3}{x-1}$

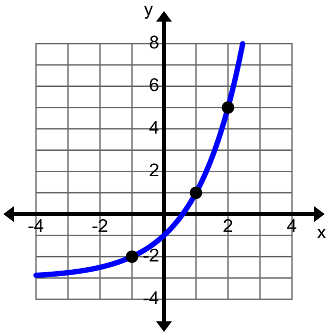
Example 5:
Use the table below to write the table for the inverse function.

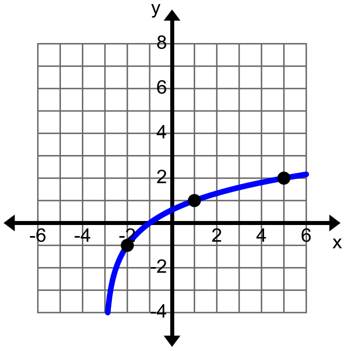
x	$f(x)=x^3-4x+1$
-7	-314
-6	-191
-5	-104
-4	-47
-3	-14
-2	1
-1	4

x	$f^{-1}(x)$
-314	-7
-191	-6
-104	-5
-47	-4
-14	-3
1	-2
4	-1

The domain of the original function becomes the range of the inverse function and the range of the original function becomes the domain of the inverse function.

Example 6:
Use the graph below to draw the graph of the inverse function.





The domain of the original function becomes the range of the inverse function and the range of the original function becomes the domain of the inverse function.

Practice Exercises C

Use the table to write the table for the inverse function.

1.

x	$f(x)$
-2	0.5
-1	1.5
0	4.5
1	13.5
2	40.5

2.

x	$f(x)$
2	3
3	4
5	5
9	6
17	7

3.

x	$f(x)$
5	1
6	3
9	4
14	5
21	6

4.

x	$f(x)$
0	4
2	2
4	-4
6	-14
8	-28

5.

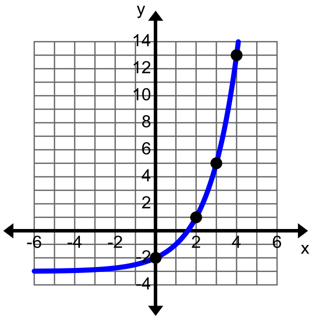
x	$f(x)$
-17	1.7
-12	1.6
-9	1.5
-7	1.4
-3	1

6.

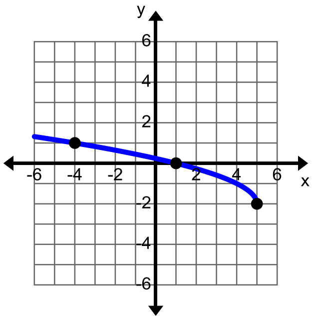
x	$f(x)$
-4	0
-3.5	1.125
-3	21
-2.5	24.375
-2	24

Use the graph to draw the graph of the inverse function.

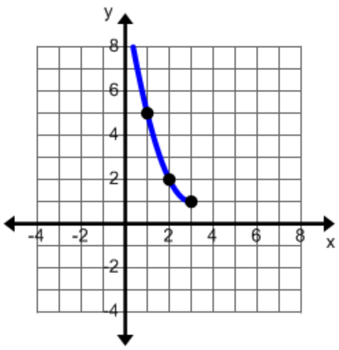
7.



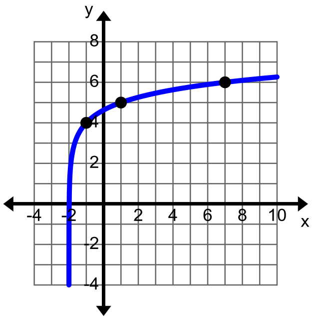
8.



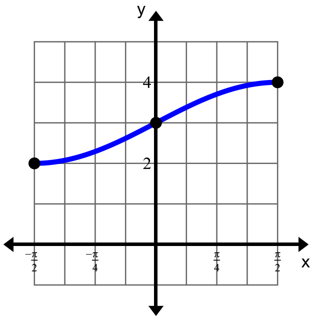
9.



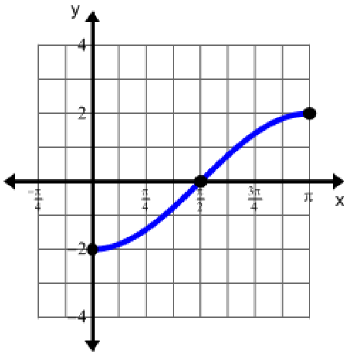
10.



11.

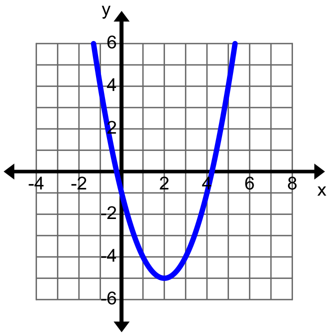


12.



Example 7:

Given $f(x) = x^2 - 4x - 1$, find a suitable domain to make this function an invertible function.

	Graph the function.
Increasing $(2, \infty)$ Decreasing $(-\infty, 2)$	Determine the intervals where the function is increasing and decreasing.
Restrict the domain to $x \ge 2$ or $x \le 2$.	Choose an interval where the function is monotonic (meaning solely increasing or solely decreasing).

Practice Exercises D

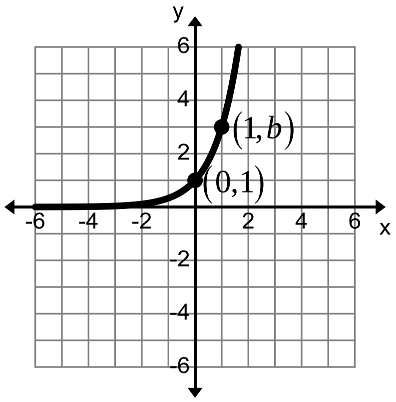
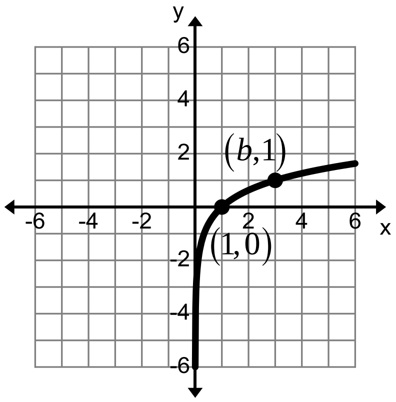
For each function, find a suitable domain to make the function an invertible function.

1. $f(x) = 2x^2 - 3$
2. $f(x) = -(x+2)^2$
3. $f(x) = (x+5)^2 + 4$
4. $f(x) = x^2 + 2x - 3$
5. $f(x) = x^2 + 12x + 32$
6. $f(x) = 2x^2 - 12x + 15$
7. $f(x) = |x-3| + 4$
8. $f(x) = -2|x| + 6$
9. $f(x) = |2-x| - 5$
10. $f(x) = -\frac{1}{2}|x+4|$
11. $f(x) = (x-7)^4 + 8$
12. $f(x) = (x+5)^4 - 3$

Unit 4 Cluster 6 (F.LE.4 and F.BF.5): Logarithms

Cluster 6: Logarithms

4.6 For exponential models, express as a logarithm the solution to a $b^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10 or *e*.
4.6 Evaluate the logarithm using technology.
4.6 Understand the inverse relationship between exponents and logarithms.
4.6 Use the relationship between exponentials and logarithms to solve problems involving logarithms and exponents.

$f(x) = b^x,\ b \neq 0,\ b \neq 1$	$f(x) = \log_b x,\ b \neq 0,\ b \neq 1$
	
Domain: $(-\infty,\infty)$ Range: $(0,\infty)$ Horizontal Asymptote: $y = 0$ Intercept: $(0,1)$ End Behavior: $\lim\limits_{x\to\infty} f(x) = \infty$; $\lim\limits_{x\to-\infty} f(x) = 0$	Domain: $(0,\infty)$ Range: $(-\infty,\infty)$ Vertical Asymptote: $x = 0$ Intercept: $(1,0)$ End Behavior: $\lim\limits_{x\to\infty} f(x) = \infty$; $\lim\limits_{x\to0^+} f(x) = -\infty$

Exponential and logarithmic functions are inverses of each other. Two of the most widely used logarithms are the common log, which is base 10, and is written as $\log_{10} x = \log x$ and the natural log, which is base *e*, and is written as $\log_e x = \ln x$.

Definition of a Logarithm

$$\log_b x = c \text{ if and only if } b^c = x$$

$$\ln x = c \text{ if and only if } e^c = x$$

Example 1:
Rewrite each of the following in exponential form.

a. $\log_4 64 = 3$

b. $\log_5 \frac{1}{25} = -2$

c. $\log_{65} 1 = 0$

a. $4^3 = 64$	The base is 4 and the exponent is 3.
b. $5^{-2} = \frac{1}{25}$	The base is 5 and the exponent is -2.
c. $65^0 = 1$	The base is 65 and exponent is 0.

Example 2:
Rewrite each of the following in logarithmic form.

a. $3^4 = 81$

b. $10^{-2} = \frac{1}{100}$

c. $6^1 = 6$

a. $\log_3 81 = 4$	The base is 3 and the exponent is 4.
b. $\log_{10} \frac{1}{100} = -2$	The base is 10 and the exponent is -2.
c. $\log_6 6 = 1$	The base is 6 and exponent is 1.

Basic Properties of Logarithms
where $b > 0$, $b \neq 1$, $x > 0$, and c is any real number

1. $\log_b 1 = 0$	1. $\ln 1 = 0$
2. $\log_b b = 1$	2. $\ln e = 1$
3. $\log_b b^c = c$	3. $\ln e^c = c$
4. $b^{\log_b x} = x$	4. $e^{\ln x} = x$

Example 3:
Use the properties of logarithms to evaluate the expression without a calculator.

a. $\log 10^{-4}$ b. $e^{\ln 6}$ c. $\log_3 1$ d. $\log_{50} 50$

a. $\log 10^{-4} = -4$	Use of basic property 3.
b. $e^{\ln 6} = 6$	Use of basic property 4.
c. $\log_3 1 = 0$	Use of basic property 1.
d. $\log_{50} 50 = 1$	Use of basic property 2.

Practice Exercises A

Rewrite each of the equations in exponential form.

1. $\log_4 1 = 0$
2. $\log_2 8 = 3$
3. $\log_9 \frac{1}{81} = -2$
4. $\log_3 243 = 5$
5. $\log_7 \frac{1}{343} = -3$
6. $\log_6 216 = 3$

Rewrite each of the equations in logarithmic form.

7. $10^{-3} = \frac{1}{1000}$
8. $5^4 = 625$
9. $6^{-1} = \frac{1}{6}$
10. $9^3 = 729$
11. $7^1 = 7$
12. $3^3 = 27$

Use the properties of logarithms to evaluate the expression without a calculator.

13. $\log_8 8$
14. $\log_9 9^3$
15. $\ln 1$
16. $4^{\log_4(2x)}$
17. $\log_5 5^{3-x}$
18. $\log_6 1$
19. $\log_9 9$
20. $e^{\ln x^2}$
21. $\ln e^{10x+5}$

The Principle of Exponential Equality

For any real number *b*, where $b \neq -1, 0,$ or 1, $b^{x_1} = b^{x_2}$ is equivalent to $x_1 = x_2$. In other words, powers of the same base are equal if and only if the exponents are equal.

Example 4:
Solve the following.

a. $\log_2 8 = x$ b. $\log_7 x = 2$ c. $\log_x 125 = 3$

a. $2^x = 8$ $2^x = 2^3$ $x = 3$	Rewrite in exponential form and solve for *x*.
b. $7^2 = x$ $49 = x$	Rewrite in exponential form and solve for *x*.
c. $x^3 = 125$ $x^3 = 5^3$ $x = 5$	Rewrite in exponential form and solve for *x*.

Example 5:
Solve the following.

a. $27^2 = 9^{x+1}$ b. $\log_4(3x-2) = 2$

a. $27^2 = 9^{x+1}$ $\left(3^3\right)^2 = \left(3^2\right)^{x+1}$ $3^6 = 3^{2x+2}$ $6 = 2x+2$ $4 = 2x$ $2 = x$	Change the bases on both sides of the equation so that they are the same base. Rewrite using exponent rules. Simplify using exponent rules. Solve using the principle of exponential equality.
b. $\log_4(3x-2) = 2$ $4^2 = 3x-2$ $16 = 3x-2$ $18 = 3x$ $6 = x$	Rewrite in exponential form and solve for *x*.

Practice Exercises B

Solve the following equations.

1. $\log_7 x = 3$
2. $\log_5 625 = x$
3. $\log_x 10{,}000 = 4$
4. $\log_x \frac{1}{225} = -2$
5. $\log_9 x = 0$
6. $\log_3 \frac{1}{243} = x$
7. $4^3 = 8^{3-x}$
8. $16^{x+1} = 8^{x+3}$
9. $\left(\frac{1}{9}\right)^{2-x} = 27^2$
10. $\log_7 (10x+3) = 3$
11. $\log_2 (17x-2) = 5$
12. $\log_4 (3x-5) = 3$

For each of the following rules, $b \neq 1$, x, y, and c are real numbers.

Product Rule

$$\log_b (xy) = \log_b x + \log_b y \qquad \ln(xy) = \ln x + \ln y$$

Quotient Rule

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y \qquad \ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

Power Rule

$$\log_b (x)^c = c \log_b x \qquad \ln(x)^c = c \ln x$$

Example 6:
Expand the following expressions.

a. $\log \frac{a^4 b}{c^5}$

b. $\ln \sqrt{m^3 n}$

c. $\log \frac{2w^4 h^3}{a^2 b^5}$

a.	
$\log \frac{a^4 b}{c^5}$	Use the product and quotient rules to rewrite the expression.
$\log a^4 + \log b - \log c^5$	
$4\log a + \log b - 5\log c$	Use the power rule to rewrite the expression.

b. $\ln\sqrt{m^3n}$ $\frac{1}{2}\ln m^3n$ $\frac{1}{2}\left(\ln m^3+\ln n\right)$ $\frac{1}{2}\left(3\ln m+\ln n\right)$ $\frac{3}{2}\ln m+\frac{1}{2}\ln n$	Use the power rule to rewrite the expression. Use the product rule to rewrite the epxression. Use the power rule to rewrite the expression. Use the distributive property.
c. $\log\frac{2w^4h^3}{a^2b^5}$ $\log 2+\log w^4+\log h^3-\left(\log a^2+\log b^5\right)$ $\log 2+4\log w+3\log h-\left(2\log a+5\log b\right)$ $\log 2+4\log w+3\log h-2\log a-5\log b$	Use the product and the quotient rules to rewrite the expression. Use the power rule to rewrite the expression. Use the distributive property.

Note: When using the quotient rule, all terms in the denominator will be subtracted.

Example 7:
Condense the following expressions.

a. $\ln(x+1)-3\ln(x-2)$ b. $\log 3+4\log a-\frac{2}{3}\log b$ c. $4\ln a-3\ln b+7\ln c-5\ln(d+1)$

a. $\ln(x+1)-3\ln(x-2)$ $\ln(x+1)-\ln(x-2)^3$ $\ln\frac{x+1}{(x-2)^3}$	Use the power rule to rewrite the expression. Use the quotient rule to rewrite the expression.

b. $\log 3+4\log a-\frac{2}{3}\log b$ $\log 3+\log a^4-\log b^{2/3}$ $\log\frac{3a^4}{\sqrt[3]{b^2}}$	Use the power rule to rewrite the expression. Use the product and quotient rules to rewrite the expression

c. $4\ln a - 3\ln b + 7\ln c - 5\ln\left(d+1\right)$ $\ln a^4 - \ln b^3 + \ln c^7 - \ln\left(d+1\right)^5$ $\ln \dfrac{a^4c^7}{b^3\left(d+1\right)^5}$	Use the power rule to rewrite the expression. Use the product and quotient rules to rewrite the expression.

Practice Exercises C

Expand the following expressions.

1. $\log_4 x^5y^7$

2. $\log_7 49xyz$

3. $\log \dfrac{a^2b^3}{c^4}$

4. $\log\left[\left(2x+1\right)\left(x+7\right)\right]$

5. $\log_8 8\sqrt{3a^5}$

6. $\log\left(\dfrac{\sqrt{x}y^3}{z^3}\right)$

7. $\log_3 \dfrac{27\left(x-3\right)}{x^2y^5}$

8. $\log_5 \sqrt[3]{\dfrac{x^2y}{25}}$

9. $\ln \dfrac{x^3\sqrt{x^2+1}}{\left(x+1\right)^4}$

Condense the following expressions.

10. $\ln x + \ln 7$

11. $\log_2 96 - \log_2 3$

12. $\log\left(2x+5\right) - \log\left(x-3\right)$

13. $4\ln x + 7\ln y - 3\ln z$

14. $5\log_6 x + 2\log_6 y - \dfrac{2}{3}\log_6 z$

15. $\dfrac{1}{2}\log_3 x + \dfrac{1}{2}\log_3 z - \dfrac{3}{2}\log_3 y$

16. $\log x + \log 7 + \log(x^2-1) - \log(x+1)$

17. $\ln\left(x-2\right) - \ln\left(x^2-4\right) - 3\ln x$

Change of Base Formula for Logarithms

Most calculators only have $\log x$ and $\ln x$. In order to evaluate logarithms with a different base, you will need the change of base formula.

$$\log_b x = \frac{\log x}{\log b},\ b \neq 1 \qquad \text{or} \qquad \log_b x = \frac{\ln x}{\ln b},\ b \neq 1$$

Example 8:
Find an approximation for the following expressions.

a. $\ln 7$ b. $\log 0.15$ c. $\log_4 17$ d. $\log_{52} 26$

a. $\ln 7 \approx 1.176$	Use the calculator.
b. $\log 0.15 \approx -0.824$	Use the calculator.
c. $\log_4 17 = \frac{\log 17}{\log 4} \approx 2.044$ $\log_4 17 = \frac{\ln 17}{\ln 4} \approx 2.044$	Use the change of base formula and your calculator.
d. $\log_{52} 26 = \frac{\log 26}{\log 52} \approx 0.825$ $\log_{52} 26 = \frac{\ln 26}{\ln 52} \approx 0.825$	Use the change of base formula and your calculator.

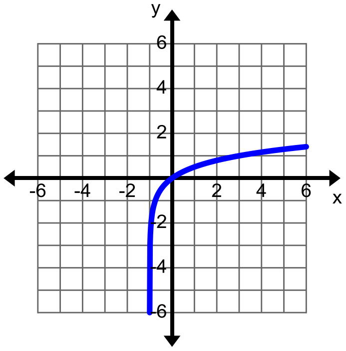
Practice Exercises D

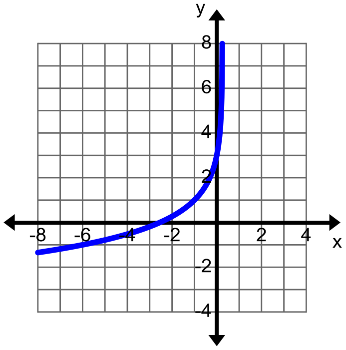
Find an approximation for the following expressions.

1. $\ln 0.5$
2. $\log 3.25$
3. $\ln 56$
4. $\log 20$
5. $\log 0.125$
6. $\log_4 200$
7. $\log_6 7780$
8. $\log_5 10$
9. $\log_3 12$
10. $\log_7 47$
11. $\log_{14} 100$
12. $\log_9 1000$

Example 9:
Find the domain of the function then graph it.

a. $f(x)=\log_4(x+1)$ b. $f(x)=-2\log_5(4x+1)+3$

a. $x+1>0$ $x>-1$ The domain is $(-1,\infty)$.	The domain of a logarithmic function has to be greater than zero.
	Use the change of base formula to enter the function. $y=\dfrac{\log(x+1)}{\log 4}$ or $y=\dfrac{\ln(x+1)}{\ln 4}$

b. $1-4x>0$ $-4x>-1$ $x<\dfrac{1}{4}$ The domain is $\left(-\infty,\dfrac{1}{4}\right)$.	The domain of a logarithmic function has to be greater than zero.
	Use the change of base formula to enter the function. $y=-2\dfrac{\log(1-4x)}{\log 5}+3$ or $y=-2\dfrac{\ln(1-4x)}{\ln 5}+3$

Practice Exercises E

Find the domain of the function and then graph it.

1.	$f(x) = \log_5(x+4)$	2.	$f(x) = \log_3(x+6)$	3.	$f(x) = \log_4(2-x)$
4.	$f(x) = \log_2(7-x)$	5.	$f(x) = \log_6(2x+1)$	6.	$f(x) = \log_7(4-3x)$
7.	$f(x) = \log_2(x-5)+2$	8.	$f(x) = -4\log_8(-2x)+7$	9.	$f(x) = \log_5(x-8)-1$
10.	$f(x) = 3\log_4(5-3x)-2$	11.	$f(x) = \log_3(5x-6)-4$	12.	$f(x) = \log_6(8x-5)+6$

The Principle of Logarithmic Equality

For any logarithmic base, *b*, and for any $x > 0$ and $y > 0$, $x = y$ is equivalent to $\log_b x = \log_b y$. In other words, two expressions are equal if and only if the logarithms of those expressions are equal.

Example 10:
Solve each equation.

a. $\log_4(x-5) = -1$

b. $50e^{0.035x} = 200$

c. $\ln(x-3) + \ln(x+4) = 3\ln(2)$

d. $25^{2x+1} = 144$

a. $\log_4(x-5) = -1$ $4^{-1} = x-5$ $\frac{1}{4} = x-5$ $\frac{21}{4} = x$	Write the equation in exponential form. Solve for *x*.
$\log_4\left(\frac{21}{4}-5\right) \overset{?}{=} -1$ $\log_4\left(\frac{1}{4}\right) \overset{?}{=} -1$ $-1 = -1$	Check your answer in the original equation.

b. $50e^{0.035x}=200$ $e^{0.035x}=4$ $\ln\left(e^{0.035x}\right)=\ln 4$ $0.035x=\ln 4$ $x=\dfrac{\ln 4}{0.035}\approx 39.608$	Isolate the exponential term. Find the natural logarithm of both sides. Use the property $\ln e^{c}=c$ to eliminate the base. Solve for *x*.

c. $\ln(x-3)+\ln(x+4)=3\ln(2)$ $\ln\left((x-3)(x+4)\right)=\ln 2^{3}$ $\ln\left(x^{2}+x-12\right)=\ln 8$ $x^{2}+x-12=8$ $x^{2}+x-20=0$ $(x+5)(x-4)=0$ $x+5=0 \qquad x-4=0$ $x=-5 \qquad x=4$	Use the product rule and the power rule to rewrite both sides of the equation. Expand both sides of the equation. Use the principal of logarithmic equality to eliminate the logarithm. Solve for *x*.
$\ln(-5-3)+\ln(-5+4)\overset{?}{=}3\ln(2)$ $\ln(-8)+\ln(-1)\overset{?}{=}3\ln(2)$ $x=-5$ is an extraneous solution. $\ln(4-3)+\ln(4+4)\overset{?}{=}3\ln(2)$ $\ln(1)+\ln(8)\overset{?}{=}3\ln(2)$ $0+\ln(8)\overset{?}{=}3\ln(2)$ $\ln\left(2^{3}\right)\overset{?}{=}3\ln(2)$ $3\ln(2)=3\ln(2)$ $x=4$ is a solution.	Check both answers in the original equation. The $\ln(-8)$ and $\ln(-1)$ are undefined.

d.	
$25^{2x+1}=144$	
$\ln\left(25^{2x+1}\right)=\ln 144$	Find the natural log of both sides.
$\left(2x+1\right)\ln 25=\ln 144$	Use the power rule to rewrite the expression.
$2x\ln 25+\ln 25=\ln 144$	Use the distributive property.
$2x\ln 25=\ln 144-\ln 25$	Solve for *x*.
$x=\dfrac{\ln 144-\ln 25}{2\ln 25}\approx 0.272$	

Practice Exercises F

Solve each equation.

1. $\log_3\left(2x-1\right)=3$
2. $\log_7\left(3x-11\right)=\log_7\left(x-3\right)$
3. $\log_6 x+\log_6 3=2$
4. $2\log_3\left(x+4\right)-\log_3 9=2$
5. $3\log_4\left(x-2\right)+\log_4 16=5$
6. $\log x+\log\left(x-21\right)=2$
7. $\log_6 x+\log_6\left(x-9\right)=2$
8. $\log_2\left(x+2\right)+\log_2\left(x+4\right)=3$
9. $\log_3\left(x+6\right)=1-\log_3\left(x+4\right)$
10. $3^x=25$
11. $2^{-x}=1.5$
12. $5^{x+3}=30$
13. $30e^{0.6x}=240$
14. $7e^{2x}=63$
15. $3\cdot 4^{2x-1}=42$
16. $4^{5-x}-2=13$
17. $5^{4x-7}-3=10$
18. $6^{3x-4}-7=65$

Example 11:
Find the inverse of each function.

a. $f\left(x\right)=\ln\left(x+2\right)-3$

b. $f\left(x\right)=\log_3\left(2x+1\right)+5$

c. $f\left(x\right)=5^{x-6}+1$

d. $f\left(x\right)=5\cdot 2^{3-x}-4$

a. $y=\ln\left(x+2\right)-3$ $x=\ln\left(y+2\right)-3$ $x+3=\ln\left(y+2\right)$ $e^{x+3}=e^{\ln\left(y+2\right)}$ $e^{x+3}=y+2$ $e^{x+3}-2=y$ $e^{x+3}-2=f^{-1}\left(x\right)$	Substitute each x with y and y with x. Isolate the logarithmic term. Use the property $e^{\ln x}=x$ to eliminate the logarithm. Solve for y.

b. $y=\log_3\left(2x+1\right)+5$ $x=\log_3\left(2y+1\right)+5$ $x-5=\log_3\left(2y+1\right)$ $3^{x-5}=3^{\log_3\left(2y+1\right)}$ $3^{x-5}=2y+1$ $3^{x-5}-1=2y$ $\dfrac{3^{x-5}-1}{2}=y$ $\dfrac{3^{x-5}-1}{2}=f^{-1}\left(x\right)$	Substitute each x with y and y with x. Isolate the logarithmic term. Use the property $b^{\log_b x}=x$ to eliminate the logarithm. Solve for y.

c. $y=5^{x-6}+1$ $x=5^{y-6}+1$ $x-1=5^{y-6}$ $\log_5\left(x-1\right)=\log_5\left(5^{y-6}\right)$ $\log_5\left(x-1\right)=y-6$ $\log_5\left(x-1\right)+6=y$ $\log_5\left(x-1\right)+6=f^{-1}\left(x\right)$	Substitute each x with y and y with x. Isolate the exponential term. Use the property $\log_b b^x=x$ to eliminate the base of the exponent. Solve for y.

d. $y = 5\cdot 2^{3-x} - 4$ $x = 5\cdot 2^{3-y} - 4$ $x + 4 = 5\cdot 2^{3-y}$ $\frac{x+4}{5} = 2^{3-y}$ $\log_2 \frac{x+4}{5} = \log_2 2^{3-y}$ $\log_2 \frac{x+4}{5} = 3 - y$ $\log_2 \frac{x+4}{5} - 3 = -y$ $-\log_2 \frac{x+4}{5} + 3 = y$ $3 - \log_2 \frac{x+4}{5} = f^{-1}(x)$	Substitute each *x* with *y* and *y* with *x*. Isolate the exponential term. Use the property $\log_b b^x = x$ to eliminate the base of the exponent. Solve for *y*.

Exercises G

Find the inverse of each function.

1. $f(x) = \log(x+7) - 2$
2. $f(x) = \log_6(x-10) - 3$
3. $f(x) = 2\ln(8-x) + 5$
4. $f(x) = \log_3(3x-4) + 1$
5. $f(x) = \log_2\left(\frac{1}{3}x + 2\right) + 7$
6. $f(x) = \log_4(1-2x) - 3$
7. $f(x) = 5^{x-3} + 2$
8. $f(x) = \frac{1}{2}e^{3-x} + 6$
9. $f(x) = 7^{x-2} - 3$
10. $f(x) = e^{4x-5} - 7$
11. $f(x) = -2\cdot 3^{5-2x} + 1$
12. $f(x) = \frac{1}{3}\cdot 2^{3x+4} - 5$

Using the Structure of Expressions to Solve Equations (Honors)

Example 12:
Solve the equation $e^{4x}-3e^{2x}-18=0$.

$e^{4x}-3e^{2x}-18=0$	
$u^2-3u-18=0$ $(u-6)(u+3)=0$ $u-6=0 \qquad u+3=0$ $u=6 \qquad u=-3$	The equation is quadratic in nature, let $u=e^{2x}$. Rewrite the equation in terms of *u*. Solve for *u*.
$e^{2x}=6 \qquad e^{2x}=-3$ $\ln e^{2x}=\ln 6$ $2x=\ln 6$ $x=\frac{\ln 6}{2}$	Substitute $e^{2x}=u$ and solve for *x*. An exponential function will never equal a negative number.

Practice Exercises H

Solve each equation.

1. $e^{2x}-2e^{x}-3=0$
2. $e^{4x}+5e^{2x}-24=0$
3. $3^{2x}+3^{x}-2=0$
4. $2^{2x}+2^{x}-12=0$
5. $e^{2x}-3e^{x}+2=0$
6. $4^{2x}+4^{x}-20=0$
7. $7^{2x}-7^{x}-30=0$
8. $e^{2x}-10e^{x}+21=0$
9. $5^{2x}+5^{x}-6=0$

Selected Answers

Secondary Mathematics 3 Answers

Unit 2 Cluster 4 (A.APR.1)

Practice Exercises A

1. $2x^2 - 3x - 8$
3. $-3x^2 + 6$
5. $2n^3 + 4n^2 - 8$
7. $-3x^2 - x + 2$
9. $9x^3 + x^2 + 6x - 10$
11. $x^2 - 4x + 10$

Practice Exercises B

1. $-30x^2 - 21x + 18$, polynomial
3. $70x^2 + 55x + 10$, polynomial
5. $-36x^2 - 77x - 40$, polynomial
7. $4x^2 + 28x + 49$, polynomial
9. $25x^6 - 10x^3 + 1$, polynomial
11. $x^3 + 2x^2 - 5x + 12$, polynomial
13. $10x^3 + 43x^2 + 30x + 7$, polynomial
15. $27x^3 + 78x^2 + 61x + 10$, polynomial
17. $x^4 + 2x^3 - x^2 - 2x - 3$, polynomial
19. $2x^4 - 6x^3 - 78x^2 - 42x - 4$, polynomial
21. $5y^4 + 13y^3 - 5y^2 - y - 12$, polynomial
23. $-20x^4 - 42x^3 - 47x^2 - 39x - 6$, polynomial

You Decide

Polynomials are closed under addition, subtraction and multiplication. All of the answers to Practice Exercises B were polynomials.

Unit 2 Cluster 5 (A.APR.2, A.APR.3, and F.IF.7c)

Practice Exercises A

1. a. $f(-1) = -24$, not a factor
   b. $f(7) = 0$, factor
   c. $f(2) = 15$, not a factor

3. a. $f(-2) = 0$, factor
   b. $f(2) = 0$, factor
   c. $f(-3) = 0$, factor

Practice Exercises B

1. $\lim_{x\to -\infty} f(x) \to -\infty$, $\lim_{x\to \infty} f(x) \to +\infty$

3. $\lim_{x\to -\infty} f(x) \to +\infty$, $\lim_{x\to \infty} f(x) \to -\infty$

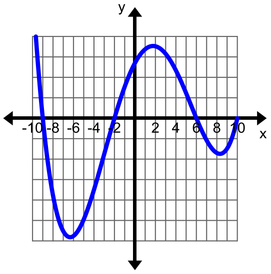
5. $\lim_{x\to -\infty} f(x) \to -\infty$, $\lim_{x\to \infty} f(x) \to -\infty$

Practice Exercises C

1. 4
3. 5
5. 6

Practice Exercises D

1.

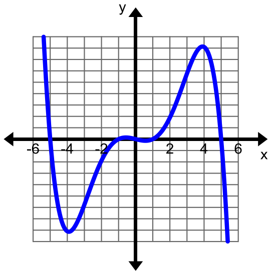


$(-9,0)$, $(-2,0)$, $(6,0)$, $(10,0)$

$\lim_{x\to -\infty} f(x) \to +\infty$

$\lim_{x\to \infty} f(x) \to +\infty$

3.

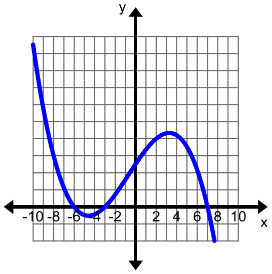


$(-5,0)$, $(-1,0)$, $(0,0)$, $(1,0)$, $(5,0)$

$\lim_{x\to -\infty} f(x) \to +\infty$

$\lim_{x\to \infty} f(x) \to -\infty$

5.

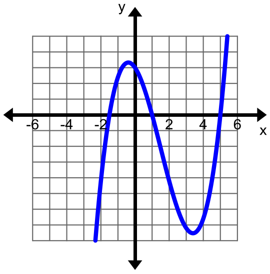


$(-6,0)$, $(-3,0)$, $(7,0)$

$\lim_{x\to -\infty} f(x) \to +\infty$

$\lim_{x\to \infty} f(x) \to -\infty$

7.



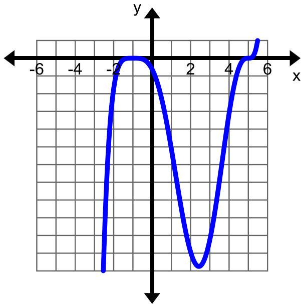
$(-1.5,0)$, $(1,0)$, $(5,0)$

$\lim_{x\to-\infty} f(x) \to -\infty$

$\lim_{x\to\infty} f(x) \to +\infty$

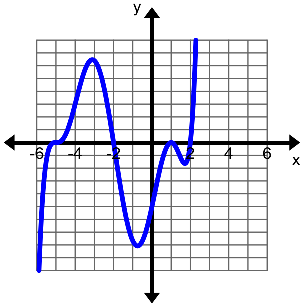
Practice Exercises E

1.



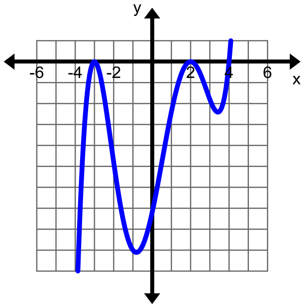
Zero	Multiplicity	Touch/Cross
$(-1,0)$	4	touch
$(5,0)$	5	cross

3.



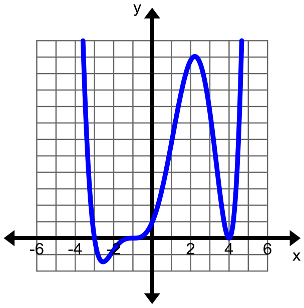
Zero	Multiplicity	Touch/Cross
$(2,0)$	1	Cross
$(-2,0)$	1	Cross
$(-5,0)$	3	Cross
$(1,0)$	2	Touch

5.



Zero	Multiplicity	Touch/Cross
$(2,0)$	2	Touch
$(-3,0)$	2	Touch
$(4,0)$	1	Cross

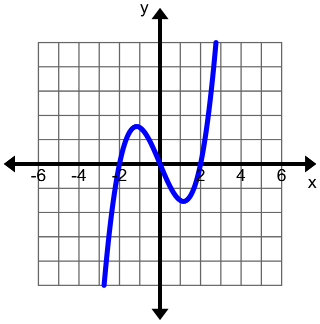
7.



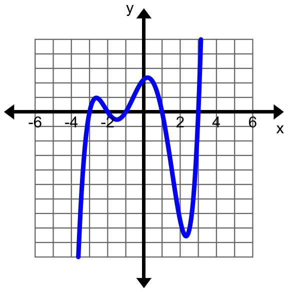
Zero	Multiplicity	Touch/Cross
$(4,0)$	2	Touch
$(-1,0)$	3	Cross
$(-3,0)$	1	Cross

Practice Exercises F

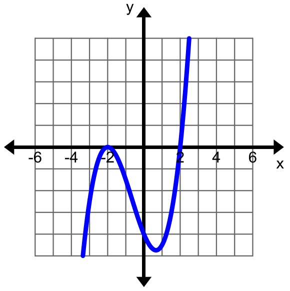
1.



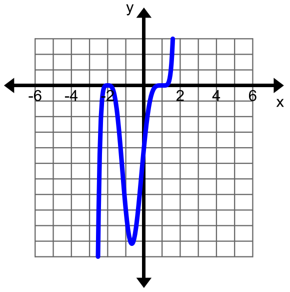
3.



5.



7.



Unit 2 Cluster 6 (A.APR.4, A.APR.5, and N.CN.8)

Practice Exercises A

1. $x^2 - x - 20$
3. $64x^3 - 48x^2y + 12xy^2 - y^3$
5. $8x^3 - 60x^2 + 150x - 125$
7. $81x^2 - 64y^2$
9. $x^2 - 16x + 39$
11. $169x^2 + 64$
13. $100x^2 + 16$
15. $x^3 - 1331y^3$
17. $16x^2 - 121$

Practice Exercises B

1. $(3x - y)(9x^2 + 3xy + y^2)$
3. $(3x - 2)^3$
5. $(3x - 8i)(3x + 8i)$
7. $(x + 11)(x + 8)$
9. $(7x + 2y)(49x^2 + 14xy + 4y^2)$
11. $(x - 5)^3$
13. $(12x - 5i)(12x + 5i)$
15. $(x + 9)(x - 5)$
17. $(x - 3)(x - 6)$
19. $x = \frac{5 + \sqrt{37}}{2}, \frac{5 - \sqrt{37}}{2}$
21. $x = 2, -1$
23. $x = -\frac{1}{3}, -2$

Practice Exercises C

1. $(x - 2 + i)(x - 2 - i)$
3. $(x + 2 + 2i)(x + 2 - 2i)$
5. $\left(x + 2 - \sqrt{3}i\right)\left(x + 2 - \sqrt{3}i\right)$
7. $\left(x + \sqrt{6}i\right)\left(x - \sqrt{6}i\right)$
9. $(x + 3)\left(x - \frac{3}{2} + \frac{3\sqrt{3}i}{2}\right)\left(x - \frac{3}{2} - \frac{3\sqrt{3}i}{2}\right)$
11. $(x + 1)\left(x - 1 + \sqrt{3}i\right)\left(x - 1 - \sqrt{3}i\right)$
13. $(x + 2)(x + 3)(x + 2i)(x - 2i)$
15. $\left(x - 1 + \sqrt{2}i\right)\left(x - 1 - \sqrt{2}i\right)(x + i)(x - i)$

Practice Exercises D

1. $(x + 3)^6 = x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729$
3. $(2x - 1)^5 = 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$
5. $(4x - 3y)^5 = 1024x^5 - 3840x^4y + 5760x^3y^2 - 4320x^2y^3 + 1620xy^4 - 243y^5$

Unit 2 Cluster 3 (A.SSE.4 and Honors)

Practice Exercises A

1. $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}=\frac{1769}{3600}$

3. $1+3+7+15+31=62$

5. $\frac{3}{5}+\frac{4}{6}+\frac{5}{7}+\frac{3}{4}+\frac{7}{9}=\frac{4421}{1260}$

7. $\sum_{k=1}^{7}2k+3$

9. $\sum_{k=1}^{10}\frac{1}{2k}$

11. $\sum_{k=1}^{9}40-3k$

Practice Exercises B

1. $\frac{1\left(1-4^7\right)}{1-4}=5461$

3. $\frac{4\left(1-(-3)^8\right)}{1-(-3)}=-6560$

5. $\frac{1\left(1-\left(\frac{1}{5}\right)^6\right)}{1-\frac{1}{5}}=\frac{3096}{3125}$

7. $\frac{-2\left(1-(-3)^5\right)}{1-(-3)}=-122$

9. $\frac{-2\left(1-6^5\right)}{1-6}=-3110$

11. $\frac{3\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}}=\frac{189}{32}$

13. \$15,304,304

15. \$1,260,008

Practice Exercises C (Honors)

1. $r=10$, diverges

3. $r=\frac{1}{4}$, converges

5. $r=1.02$, diverges

7. $S=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$

9. $S=\frac{2}{1-\left(-\frac{1}{5}\right)}=\frac{5}{3}$

11. $S=\frac{12}{1-\frac{1}{2}}=24$

13. $S=\frac{3}{1-0.1}=\frac{10}{3}$

15. $S=\frac{1}{1-\frac{e}{3}}=\frac{3}{3-e}$

17. $S=\frac{-2}{1-0.6}=-5$

19. $4+2\sum_{k=1}^{\infty}3.92(0.98)^{k-1}=4+2\left(\frac{3.92}{1-0.98}\right)$

396 feet

Unit 2 Cluster 7 (A.APR.6, and A.APR.7)

Practice Exercises A

1. $4x^2-2x-1$

3. $3x+4+\frac{9}{x}$

5. $x-5$

7. $6x+1$

9. $\frac{1}{2x-7}$

11. $\frac{x+7}{x-6}$

13. $\frac{x-7}{x+10}$

15. $\frac{3x-2}{x+1}$

17. $\frac{1}{x^2-2x+4}$

19. $\frac{x-4}{x^2-4x+16}$

21. $\frac{1}{2x+1}$

Practice Exercises B

1. $x-5-\frac{50}{x-5}$

3. $x-5+\frac{1}{x-4}$

5. $4x+2+\frac{5}{x-1}$

7. $2x-1+\frac{1}{3x-2}$

9. $x^2-x+1-\frac{2}{x+1}$

11. $2x^2-x+1-\frac{5}{x+2}$

13. $3x^2-x+4+\frac{10}{2x-3}$

15. $3x^2+x-1-\frac{4}{x-2}$

Practice Exercises C

1. $x^2+2x-3-\frac{12}{-5x^2+9x-2}$

3. $x^2+3x-\frac{13}{x^2+5x+2}$

5. $2x^2-\frac{5}{4x^2-x-9}$

Practice Exercises D

1. $\frac{4y}{3x^3}$, rational expression
3. $2x^5$, rational expression
5. $\frac{8x^4}{5}$, rational expression
7. $\frac{2}{x-5}$, rational expression
9. $\frac{2}{3x}$, rational expression
11. $\frac{2}{x}$, rational expression
13. $\frac{(x-2)(3x+2)}{(x+1)(2x+1)}$, rational expression
15. $\frac{x+2}{4x(x+6)}$, rational expression
17. $\frac{5x^3}{x-5}$, rational expression
19. $\frac{(x+2)(x+5)}{3(x-1)}$, rational expression
21. $\frac{x^2+4x+16}{(x+4)(x+4)}$, rational expression
23. $\frac{x(5x-3)}{5(x-4)}$, rational expression
25. $\frac{15}{4x+3}$, rational expression

YOU DECIDE

Rational functions are closed under multiplication and division because all of the answers in Exercises D were rational functions.

Practice Exercises E

1. $\frac{2}{x}$, rational expression
3. 2, rational expression
5. $\frac{1}{5x+7}$, rational expression
7. $\frac{12-x}{2(x+1)}$, rational expression
9. $\frac{11}{6(x-5)}$, rational expression
11. $\frac{-3}{x-1}$, rational expression
13. $\frac{1}{x+7}$, rational expression
15. $\frac{x^2+14x-49}{x^2-49}$, rational expression
17. $\frac{-x-3}{(x-3)(x-2)}$, rational expression
19. $\frac{5x+5}{(x-2)(x+2)(x+3)}$, rational expression
21. $\frac{40}{(x-5)(x+5)(x+5)}$, rational expression
23. $\frac{13x-4}{(x-2)(2x+3)}$, rational expression
25. $\frac{-2x^2-3x-4}{x(x-1)(x+2)}$, rational expression

YOU DECIDE

Rational functions are closed under addition and subtraction because all of the answers in Exercises E were rational functions.

Unit 2 Cluster 8 (A.REI.2)

Practice Exercises A

1. $x \neq 9$
3. $x \neq -6$ or $x \neq 1$
5. $x \neq 0$
7. $x \neq 0$ or $x \neq 1$
9. $x \neq -\frac{2}{3}$ or $x \neq 10$

Practice Exercises B

1. $x = 12$ or $x = -1$
3. $x = -9$
5. $x = -7$
7. $x = -4$
9. $x = \frac{11}{7}$
11. no solution
13. $x = -3$
15. no solution
17. $x = 2$ or $x = 3$
19. no solution
21. $x = -2$ or $x = 6$
23. $x = -\frac{10}{3}$

Practice Exercises C

1. $x = 11$
3. $x = -3$
5. $x = 13$
7. $x = 25$
9. $x = 15$
11. $x = 33$
13. $x = 20$
15. No solution
17. $x = -1$
19. $x = -125$
21. $x = -5$
23. $x = 9$
25. $x = 7$ or $x = 3$
27. $x = 2$ or $x = -1$

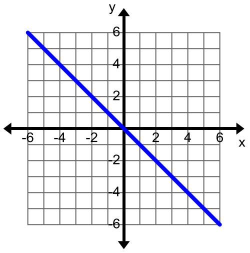
Practice Exercises D

1. $x = 1,\ x = -15$
3. $x = 20$
5. $x = 1$
7. $x = 6$
9. $x = -12,\ x = -28$

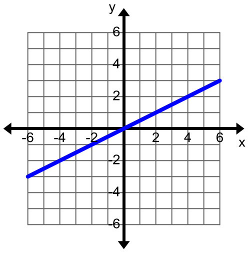
Unit 4 Clusters 3 and 5 (F.IF.7b,e and F.BF.3)

Practice Exercises A

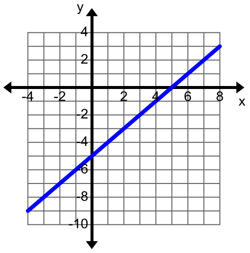
1a.



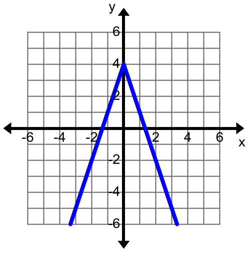
1b.



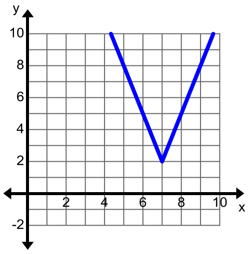
1c.



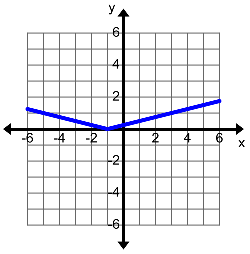
3a.



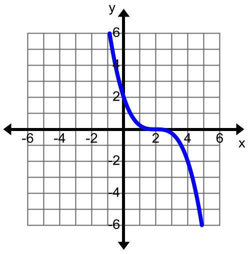
3b.



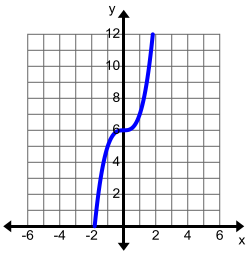
3c.



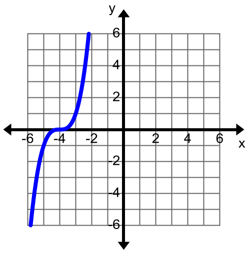
5a.



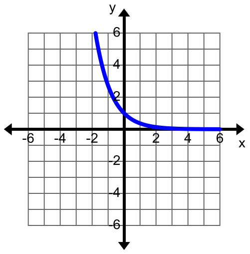
5b.



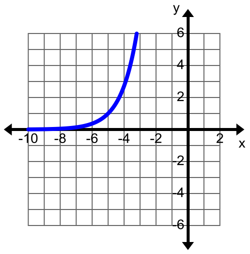
5c.



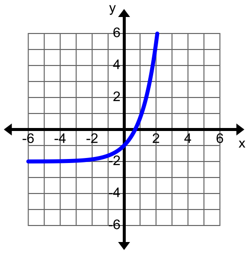
7a.



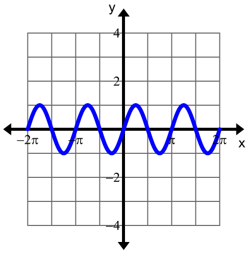
7b.



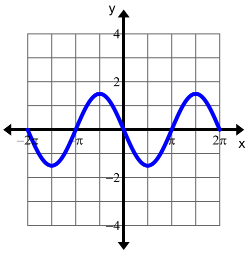
7c.



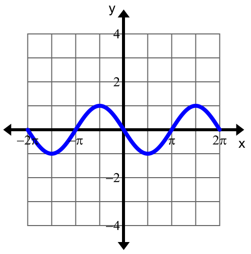
9a.



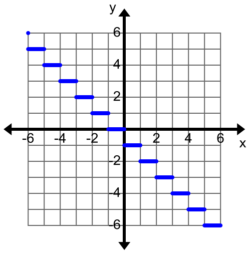
9b.



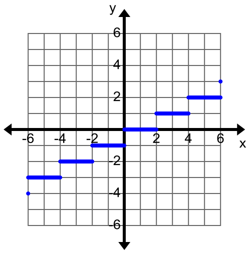
9c.



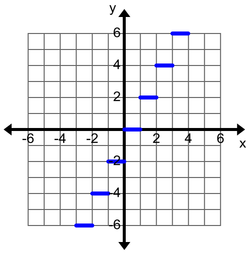
11a.



11b.



11c.



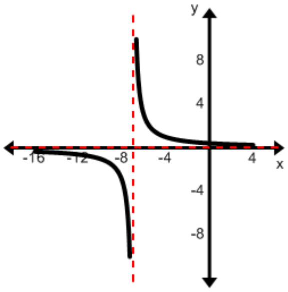
Practice Exercises B

1. yes, translated down 4
3. yes, stretched vertically by a factor of 4, translated down 4 units and left 0.5 units.
5. yes, reflected over the *x*-axis, stretched vertically by a factor of 2
7. yes, translated 3 units left and down 5 units
9. no
11. yes, translated 1 unit left
13. reflected over the *x*-axis, stretched vertically by a factor of 2, translated 3 units left and up 4 units
15. reflected over the *x*-axis, translated 3 units to the right and down 2 units
17. reflected over the *x*-axis, stretched vertically by a factor of 4, translated 2 units to the right

Unit 4 Cluster 3 HONORS (F.IF.7d)

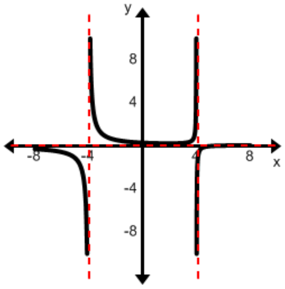
Practice Exercises A

1.



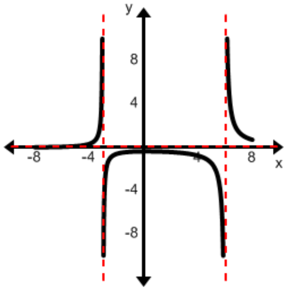
D: $(-\infty,-7)\cup(-7,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

3.



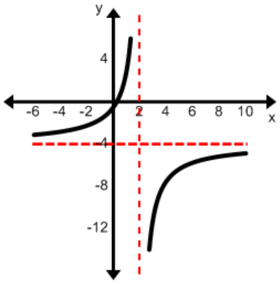
D: $(-\infty,-4)\cup(-4,4)\cup(4,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

5.



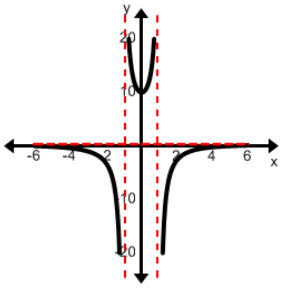
D: $(-\infty,-3)\cup(-3,6)\cup(6,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

7.



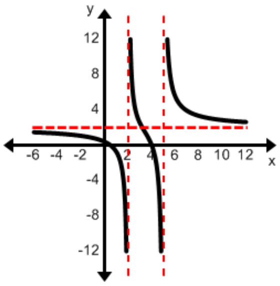
D: $(-\infty,2)\cup(2,\infty)$; R: $(-\infty,-4)\cup(-4,\infty)$

9.



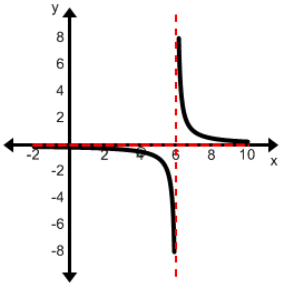
D: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$; R: $(-\infty,0)\cup(10,\infty)$

11.



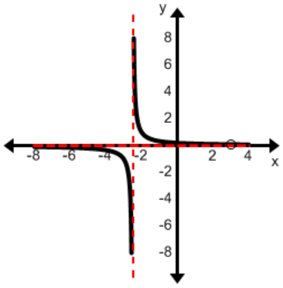
D: $(-\infty,2)\cup(2,5)\cup(5,\infty)$; R: $(-\infty,\infty)$

13.



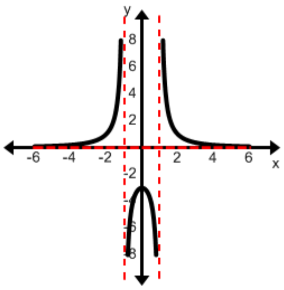
D: $(-\infty,4)\cup(4,6)\cup(6,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

15.



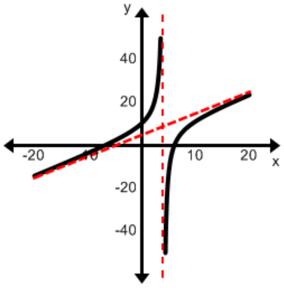
D: $\left(-\infty,-\frac{5}{2}\right)\cup\left(-\frac{5}{2},3\right)\cup(3,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

17.



D: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

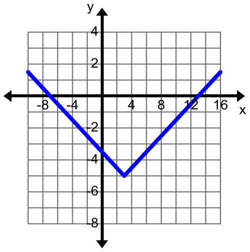
19.



D: $(-\infty,4)\cup(4,\infty)$

Unit 4 Cluster 2 (F.IF.4, and F.IF.5)
Practice Exercises A

1a.



1b. $(-7,0)$, $(13,0)$, and $\left(0,-\frac{3}{5}\right)$

1c. Minimum: $(3,-5)$

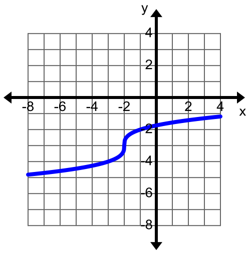
1d. Increasing: $(3,\infty)$, decreasing: $(-\infty,3)$

1e. Positive: $(-\infty,-7)\cup(13,\infty)$, negative: $(-7,13)$

1f. $\lim_{x\to\infty} f(x)=\infty$, $\lim_{x\to-\infty} f(x)=\infty$

1g. No symmetry

3a.



3b. $(25,0)$ and $(0,-1.740)$

3c. none

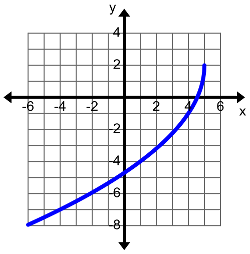
3d. Increasing: $(-\infty,\infty)$

3e. Positive: $(25,\infty)$, negative: $(-\infty,25)$

3f. $\lim_{x\to\infty} f(x)=\infty$, $\lim_{x\to-\infty} f(x)=-\infty$

3g. No symmetry

5a.



5b. $\left(\frac{41}{9},0\right)$ and $(0,-3.708)$

5c. Maximum: $(5,2)$

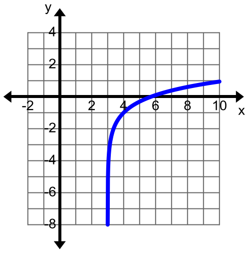
5d. Increasing: $(-\infty,5)$

5e. Positive: $\left(\frac{41}{9},5\right)$, negative: $\left(-\infty,\frac{41}{9}\right)$

5f. $\lim_{x\to5^-} f(x)=2$, $\lim_{x\to-\infty} f(x)=-\infty$

5g. No symmetry

7a.



7b. $(6,0)$

7c. None

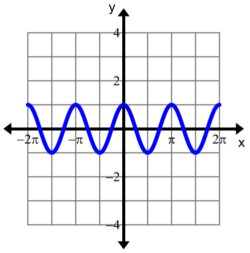
7d. Increasing: $(3,\infty)$

7e. Positive: $(6,\infty)$, negative: $(3,6)$

7f. $\lim_{x\to\infty} f(x)=\infty$, $\lim_{x\to3^+} f(x)=-\infty$

7g. No symmetry

9a.



9b. $\left(\pm\frac{\pi}{4}k,0\right)$ where k is odd, and $(0,1)$

9c. Absoluate maximum of 1 and absolute minimum of -1

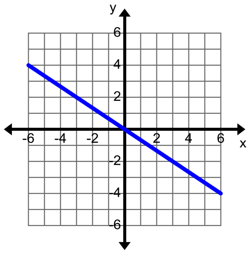
9d. alternating increasing and decreasing in periodic waves

9e. alternating positive and negative in periodic waves

9f. no end behavior because the values oscillate between -3 and 3 and approach no limit

9g. Even symmetry

11a.



11b. $(0,0)$

11c. None

11d. Decreasing: $(-\infty,\infty)$

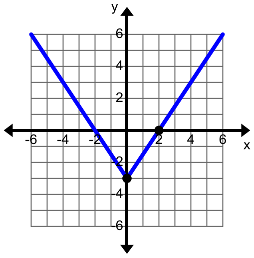
11e. Positive: $(-\infty,0)$, negative: $(0,\infty)$

11f. $\lim\limits_{x\to\infty} f(x) = -\infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$

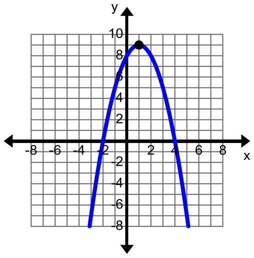
11g. Odd symmetry

Practice Exercises B

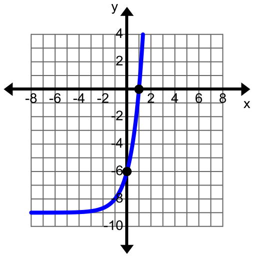
1.



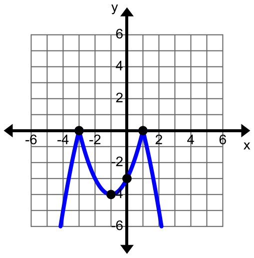
3.



5.



7.



Practice Exercises C

1. $(-\infty,-4)\cup(-4,4)\cup(4,\infty)$
3. $(-\infty,\infty)$
5. $(-\infty,\infty)$
7. $(-\infty,\infty)$
9. $(-\infty,\infty)$
11. $(-\infty,-5)\cup(-5,2)\cup(2,\infty)$
13. $(-\infty,\infty)$
15. $(-\infty,\infty)$
17. $(-\infty,\infty)$
19. $(0,12)$
21. $(0,7.5)$

Unit 4 Cluster 3 (F.IF.9)

Practice Exercises A

1. **Function A**
   - a. increasing $(1,\infty)$
   - b. positive $(1,\infty)$
   - c. minimum $(1,4)$
   - d. domain $[1,\infty)$; range $[4,\infty)$

   Function B
   - a. decreasing $(1,\infty)$
   - b. positive $(1,17)$; negative $(17,\infty)$
   - c. maximum $(1,4)$
   - d. domain $[1,\infty)$; range $(-\infty,4]$

Both functions have the same domain. Function A is increasing on its entire domain while Function B is decreasing on its entire domain. Function B is a reflection of Function A over the line $y=4$.

3. **Function A**
   - a. intercepts $(-6,0),(0,0),(6,0)$
   - b. maximum $(-3,4)$; minimum $(3,-4)$
   - c. range $[-4,4]$
   - d. odd symmetry

   Function B
   - a. intercepts $(-6,0),(0,0),(6,0)$
   - b. maximum $(3,4)$; minimum $(-3,-4)$
   - c. range $[-4,4]$
   - d. odd symmetry

Both functions have the same intercepts, the same range, and the same symmetry. The maximums and minimums occur at different places. Function B is a reflection of function A over the *y*-axis or the *x*-axis.

5. **Function A**
   a. intercepts $(-1,0),(3,0),(0,3)$
   b. increasing $(-\infty,1)$; decreasing $(1,\infty)$
   c. maximum $(1,6)$
   d. domain $(-\infty,\infty)$; range $(-\infty,6]$
   Function B
   a. intercepts $(0,0),(2,0)$
   b. increasing $(-\infty,1)$; decreasing $(1,\infty)$
   c. maximum $(1,3)$
   d. domain $(-\infty,\infty)$; range $(-\infty,3]$

Both functions are increasing and decreasing on the same intervals. Both functions have their maximums occur at the same place, but the maximum value of Function A is three more than Function B. Function B is Function A shifted down three units.

7. The rocket that is graphed is in the air longer. The graphed rocket will also have the greatest maximum because it is in the air the longest.

Unit 4 Cluster 2 (F.IF.6)

Practice Exercises A

1. 5
3. $-a-3$
5. 1
7. $\frac{1}{5}$
9. $\frac{2}{5}$
11. $\frac{3}{5}$
13. 8
15. 5
17. 0.9242
19. $\frac{8}{3\pi}$
21. $-\frac{6}{\pi}$
23. -23
25. the average temperature increased 3.428 $^\circ F$ each month from March to October
27. the percentage of the labor force in unions decreased by 0.53 percent each year from 1975 to 1995.
29. the height of baby boys increased 0.5 inches per month from 6 to 16 months.
31. The liquid decreased in temperature 4.375 degrees Fahrenheit per minute for 4 to 12 minutes.

Unit 2 Cluster 9 (A.REI.11)

Practice Exercises A

1. at $x=-1$ and $x=2$
3. at $x=-2$ and $x=3$

Practice Exercises B

1. $x=0.209$ and $x=4.791$
3. $x=-0.679$ and $x=1.179$
5. $x=-0.484$ and $x=4.776$
7. $x=2$, $x=6.382$ and $x=8.618$
9. $x=-1$ and $x=5.372$
11. $x=0.712$ and $x=5.41$
13. $x=14.186$ and $x=-15.048$
15. $x=-10.181$ and $x=-2.146$

Unit 4 Cluster 1 (A.CED.1, A.SSE.2, and A.CED.4)

Practice Exercises A

1. 3 inches
3. youngest 9, middle 10, oldest 12
5. 2.4 hours
7. 12 hours
9. 82.645 psi

Practice Exercises B

1. $x=3$, $x=-7$
3. $x=\frac{8}{3}$, $x=3$
5. $x=1$
7. $x=-\frac{3}{2}$, $x=-3$
9. $x=-\frac{2}{3}$, $x=-\frac{4}{3}$
11. $x=-\frac{1}{8}$, $x=27$
13. $x=-\frac{1}{3}\sqrt{\frac{1}{3}}$, $x=\frac{1}{3}\sqrt{\frac{1}{3}}$
15. $x=0$, $x=1$
17. $x=4$, $x=0$, $x=-4$
19. $x=0$, $x=6$, $x=-3$
21. $x=3$, $x=1$, $x=-1$

Practice Exercises C

1. $(-\infty,-4]\cup[3,\infty)$
3. $(-\infty,-4]\cup[1,\infty)$
5. $(-\infty,-5)\cup(0,3)$
7. $(-\infty,-1]\cup[0,7]$
9. $(-\infty,-3)\cup[0,\infty)$
11. $(-\infty,-3)\cup[-2,3)$
13. $(0,2)\cup(2,\infty)$
15. $\left[\frac{4}{3},\infty\right)$
17. $(0,1.343)$ seconds
19. $(0,1.5]\cup[2.628,7)$ inches
21. $(0,8.919)$ centimeters

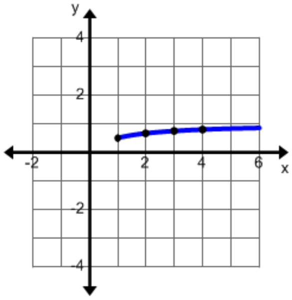
Practice Exercises D

1. $B=\pm\sqrt{\dfrac{2Vm}{qr^2}}$
3. $c=\dfrac{(2ax+b)-b^2}{-4a}$
5. $y=\pm\sqrt{b^2-\dfrac{b^2x^2}{a^2}}$
7. $b=\pm\sqrt{k^2+4ac}$
9. $a_n=\dfrac{2S}{n}-a_1$
11. $r=\sqrt[3]{\dfrac{3V}{4\pi}}$
13. $r_2=\dfrac{Rr_1}{r_1-R}$
15. $y_2=y_1\pm\sqrt{d^2-(x_2-x_1)^2}$

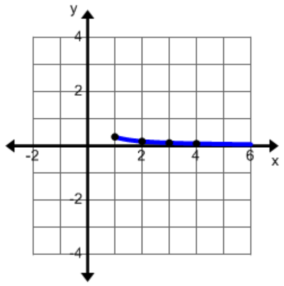
Unit 4 Cluster 1 (A.CED.2 and A.CED.3)

Practice Exercises A

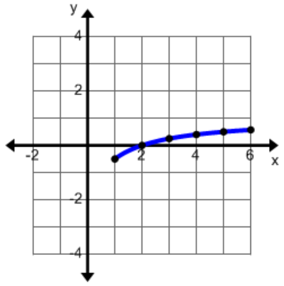
1. $f(x)=\dfrac{x}{x+1}$ for $x\ge1$



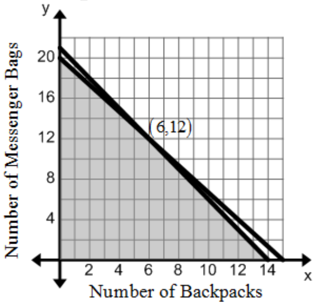
3. $f(x)=\dfrac{1}{3x}$ for $x\ge1$



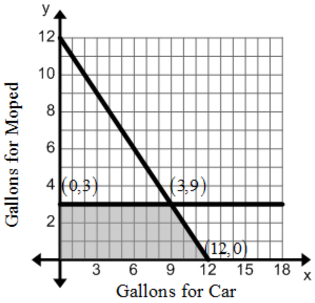
5. $f(x)=\dfrac{x-2}{x+1}$ for $x\ge1$



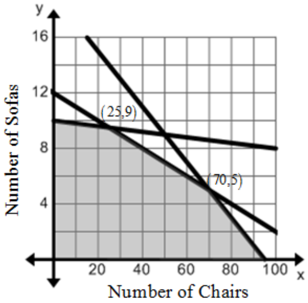
7. 6 backpacks and 12 messenger bags



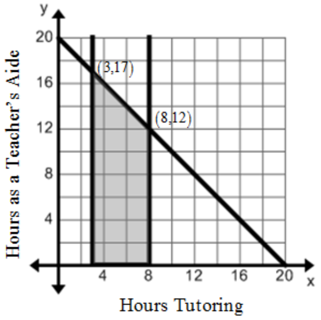
9. 9 gallons for the car, 3 gallons for the moped; 480 miles



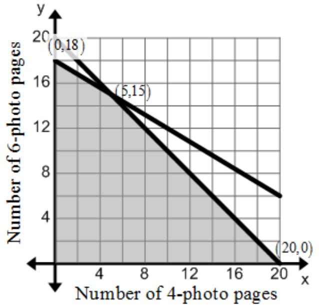
11. 25 chairs and 9 sofas



13. 8 hours tutoring, 12 hours as a teacher's aide.



15. 5 of 4 photo pages, 15 of 6 photo pages; 110 photos



Unit 4 Cluster 4 (F.BF.1, F.BF.1c honors)

Practice Exercises A

1a. $h(x)=\sqrt{x-4}+2-3x^2$; $[4,\infty)$

1b. $h(x)=\sqrt{x-4}+2+3x^2$; $[4,\infty)$

1c. $h(x)=-3x^2\sqrt{x-4}-6x^2$; $[4,\infty)$

1d. $h(x)=\dfrac{\sqrt{x-4}}{-3x^2}$; $[4,\infty)$

3a. $h(x)=\sin x+x^3-3$; $(-\infty,\infty)$

3b. $h(x)=\sin x-x^3+3$; $(-\infty,\infty)$

3c. $h(x)=x^3\sin x-3\sin x$; $(-\infty,\infty)$

3d. $h(x)=\dfrac{\sin x}{x^3-3}$; $\left(-\infty,\sqrt[3]{3}\right)\cup\left(\sqrt[3]{3},\infty\right)$

5a. $h(x)=x^3-7x+6$; $(-\infty,\infty)$

5b. $h(x)=x^3-2x^2+7x-6$; $(-\infty,\infty)$

5c. $h(x)=x^5-8x^4+13x^3-6x^2$; $(-\infty,\infty)$

5d. $h(x)=\dfrac{x^3-x^2}{x^2-7x+6}$; $(-\infty,1)\cup(1,6)\cup(6,\infty)$

7a. $h(x)=\cos(3x)+\sqrt[3]{x+1}$; $(-\infty,\infty)$

7b. $h(x)=\cos(3x)-\sqrt[3]{x+1}$; $(-\infty,\infty)$

7c. $h(x)=\cos(3x)\sqrt[3]{x+1}$; $(-\infty,\infty)$

7d. $h(x)=\dfrac{\cos(3x)}{\sqrt[3]{x+1}}$; $(-\infty,-1)\cup(-1,\infty)$

Practice Exercises B

1. -5
3. 4
5. 4
7. 0
9. 3
11. 3

13. $P(x)=0.75x^2+50x-19{,}900$; \$750,049,980,100

15. $\overline{C}(x)=0.5x^2-34x+1213$; \$2813

Practice Exercises C

1a. $h(x)=\dfrac{-8x+57}{x-7}$; $(-\infty,7)\cup(7,\infty)$

1b. $h(x)=\dfrac{1}{x-15}$; $(-\infty,15)\cup(15,\infty)$

1c. $h(x)=x-16$; $(-\infty,\infty)$

1d. $h(x)=\dfrac{-x+7}{7x-50}$; $(-\infty,7)\cup\left(7,\frac{50}{7}\right)\cup\left(\frac{50}{7},\infty\right)$

3a. $h(x)=\sqrt{x^2-9}$; $(-\infty,-3)\cup(3,\infty)$

3b $h(x)=x-9$; $[6,\infty)$

3c. $h(x)=\sqrt{\sqrt{x-6}-6}$; $[42,\infty)$

3d. $h(x)=x^4-6x^2+6$; $(-\infty,\infty)$

5a. $h(x)=x$; $(-\infty,\infty)$

5b. $h(x)=x$; $(-\infty,\infty)$

5c. $h(x)=\sqrt[3]{\sqrt[3]{x-2}-2}$; $(-\infty,\infty)$

5d. $h(x)=x^9+6x^6+12x^3+10$; $(-\infty,\infty)$

7a. $h(x)=\cos(4-x)$; $(-\infty,\infty)$

7b. $h(x)=4-\cos(x)$; $(-\infty,\infty)$

7c. $h(x)=\cos(\cos(x))$; $(-\infty,\infty)$

7d. $h(x)=x$; $(-\infty,\infty)$

Practice Exercises D

1. 9
3. 3
5. 1
7. 19
9. $\frac{7}{13}$
11. It is the same amount whether you apply the coupon or the tax first.

Unit 4 Cluster 5 (F.BF.4, F. BF.4b,c,d honors)

Practice Exercises A

1. $f^{-1}(x)=\dfrac{x-8}{-6}$
3. $f^{-1}(x)=\dfrac{x-5}{3}$
5. $f^{-1}(x)=2+\sqrt{x+16}$
7. $f^{-1}(x)=x^2-4,\ x\geq 0$
9. $f^{-1}(x)=-\dfrac{1}{4}x^2+3,\ x\geq 0$
11. $f^{-1}(x)=\dfrac{x+5}{x-3}$
13. $f^{-1}(x)=\dfrac{-2x-6}{3x-7}$
15. $f^{-1}(x)=\sqrt[3]{2x+6}$
17. $f^{-1}(x)=\sqrt[3]{x-5}+2$
19. $f^{-1}(x)=\left(\dfrac{x-7}{-2}\right)^3+5$
21. $f^{-1}(x)=(x+4)^3+1$

Practice Exercises B

1. $(f\circ g)=x,\ (g\circ f)=x$
3. $(f\circ g)=x,\ (g\circ f)=x$
5. $(f\circ g)=x,\ (g\circ f)=x$
7. $(f\circ g)=x,\ (g\circ f)=x$
9. $(f\circ g)=x,\ (g\circ f)=x$

Practice Exercises C

1.

x	$f^{-1}(x)$
0.5	-2
1.5	-1
4.5	0
13.5	1
2	40.5

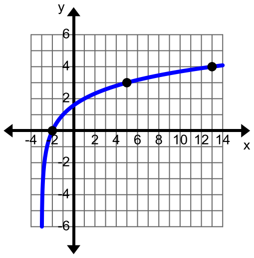
3.

x	$f^{-1}(x)$
1	5
3	6
4	9
5	14
6	21

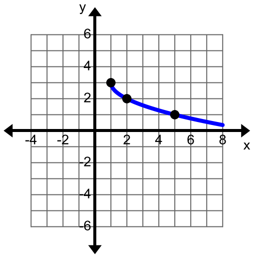
5.

x	$f^{-1}(x)$
1.7	-17
1.6	-12
1.5	-9
1.4	-7
1	-3

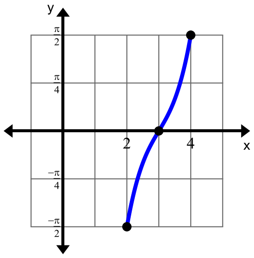
7.



9.



11.



Practice Exercises D

1. $x \ge 0$ or $x \le 0$
3. $x \ge -5$ or $x \le -5$
5. $x \ge -6$ or $x \le -6$
7. $x \ge 3$ or $x \le 3$
9. $x \ge 2$ or $x \le 2$
11. $x \ge 7$ or $x \le 7$

Unit 4 Cluster 6 (F.LE.4 and F.BF.5)

Practice Exercises A

1. $4^0 = 1$
3. $9^{-2} = \frac{1}{81}$
5. $7^{-3} = \frac{1}{343}$
7. $\log_{10} \frac{1}{1000} = -3$
9. $\log_6 \frac{1}{6} = -1$
11. $\log_7 7 = 1$
13. 1
15. 0
17. $3 - x$
19. 1
21. $10x + 5$

Practice Exercises B

1. 343
3. 10
5. 1
7. 1
9. 5
11. 2

Practice Exercises C

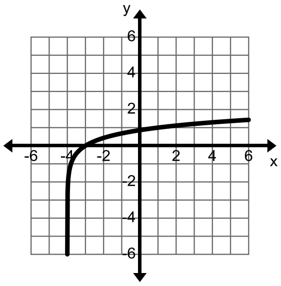
1. $5\log_4 x + 7\log_4 y$
3. $2\log a + 3\log b - 4\log c$
5. $1 + \frac{1}{2}\log_8 3 + \frac{5}{2}\log_8 a$
7. $3 + \log_3(x-3) - 2\log_3 x - 5\log_3 y$
9. $3\ln x + \frac{1}{2}\ln\left(x^2+1\right) - 4\ln(x+1)$
11. $\log_2\left(\frac{96}{3}\right) = 5$
13. $\ln\left(\frac{x^4 y^7}{z^3}\right)$
15. $\log_3 \sqrt{\frac{xz}{y^3}}$
17. $\ln \frac{x-2}{\left(x^2-4\right)x^3} = \ln \frac{1}{x^3(x+2)}$

Practice Exercises D

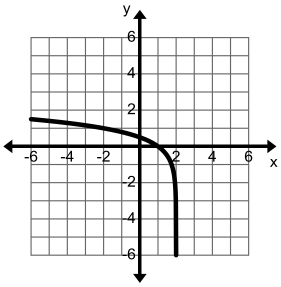
1. -0.693
3. 4.025
5. -0.903
7. 5.000
9. 2.262
11. 1.745

Practice Exercises E

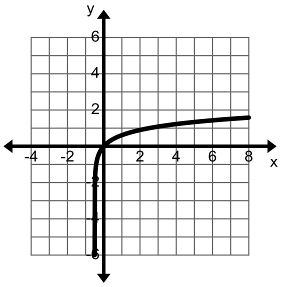
1. $(-4, \infty)$



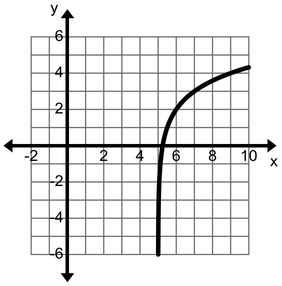
3. $(-\infty, 2)$



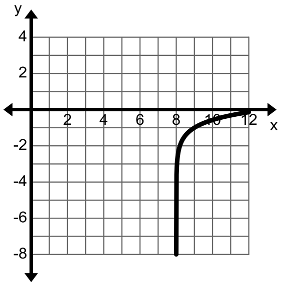
5. $\left(-\frac{1}{2}, \infty\right)$



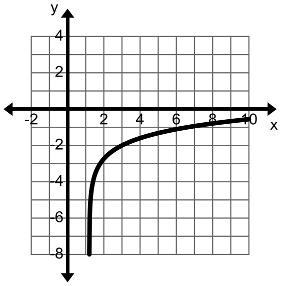
7. $(5, \infty)$



9. $(8, \infty)$



11. $\left(\frac{6}{5}, \infty\right)$



Practice Exercises F

1. $x = 5$
3. $x = 12$
5. $x = 6$
7. $x = 12$
9. $x = -3$
11. $x = -\log_2 1.5$ ≈ -0.585

Practice Exercises G

1. $f^{-1}(x) = 10^{x+2} - 7$
3. $f^{-1}(x) = 8 - e^{\frac{x-5}{2}}$
5. $f^{-1}(x) = 3 \cdot 2^{x-7} - 6$
7. $f^{-1}(x) = \log_5 (x-2) + 3$
9. $f^{-1}(x) = \log_7 (x+3) + 2$
11. $f^{-1}(x) = -\frac{1}{2}\log_3\left(-\frac{1}{2}x + \frac{1}{2}\right) + \frac{5}{2}$

Practice Exercises H

1. $x = \ln 3 \approx 1.099$
3. $x = \log_3 1 = \dfrac{\ln 1}{\ln 3} = 0$
5. $x = \ln 2 \approx 0.693,\ x = \ln 1 = 0$
7. $x = \log_7 6 = \dfrac{\ln 6}{\ln 7} \approx 0.921$
9. $x = \log_5 2 = \dfrac{\ln 2}{\ln 5} \approx 0.431$