Secondary Mathematics 3 Table of Contents

Trigonometry

Unit 3 Cluster 1 (G.SRT.9): Area of a Triangle

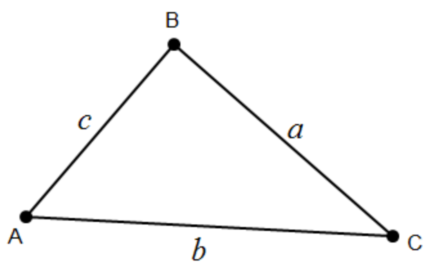
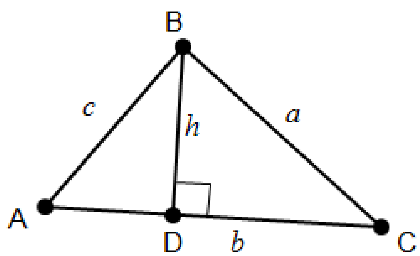
Cluster 1: Apply trigonometry to general triangles

3.1 Derive the formula for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

3.1 Find the area of triangles using the formula.

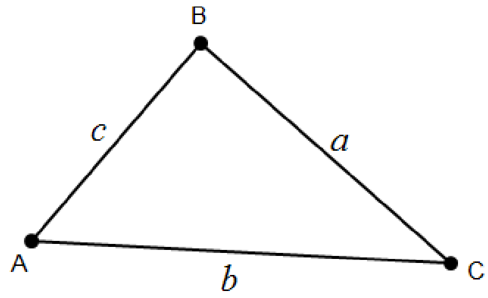
It is possible to find the area of a triangle using trigonometry when given two sides and the included angle. In order to do this, you must draw an altitude from the vertex of the non-included angle to the side opposite the angle.

Derivation of the area of a triangle

	Start with any triangle that has angles A, B, and C and side lengths, *a*, *b*, and *c*, where *a* is the side opposite angle A, *b* is the side opposite angle B, and *c* is the side opposite angle C.
	Construct an altitude, *h*, from the vertex of one of the angles to the side opposite the angle. The two triangles formed, $\triangle ABD$ and $\triangle BDC$, are right triangles.
$\sin A = \frac{h}{c}$ $c\sin A = h$	Find the measure of *h* in terms of $\angle A$ and side *c* using the sine ratio for $\angle A$.
$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$ $\text{Area} = \frac{1}{2}b(c\sin A)$ $\text{Area} = \frac{1}{2}bc(\sin A)$	The base of the entire triangle is *b* and the height is *h*. Substitute $c\sin A = h$ into the area formula for *h*, the height. NOTE: This formula works for acute, obtuse, and right triangles.

Area of a Triangle Given Two Sides and the Included Angle

The area of a triangle is half the product of the lengths of two sides and the sine of the included angle.



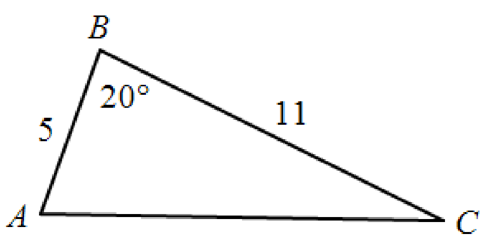
$$\text{Area} = \frac{1}{2}bc(\sin A)$$

$$\text{Area} = \frac{1}{2}ac(\sin B)$$

$$\text{Area} = \frac{1}{2}ab(\sin C)$$

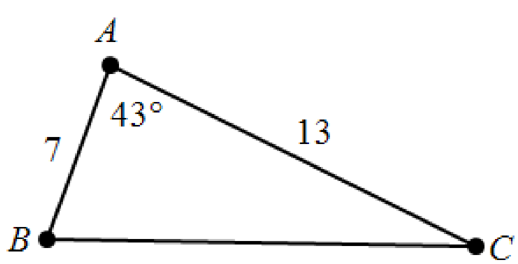
Example 1:

Find the area of a triangle with sides a = 11, b = 5, and $m\angle C = 20°$. Round your answer to the nearest thousandth (3 decimal places).

	Draw the triangle.
$\text{Area} = \frac{1}{2}ab(\sin C)$ $\text{Area} = \frac{1}{2}(11)(5)(\sin 20°)$ $\text{Area} = \frac{55}{2}(\sin 20°) \approx 9.406$	Use the formula for the area of a triangle. Substitute in known values and simplify.

Example 2:

Find the area of a triangle with sides b = 13, c = 7, and $m\angle A = 43°$. Round your answer to the nearest thousandth (3 decimal places).

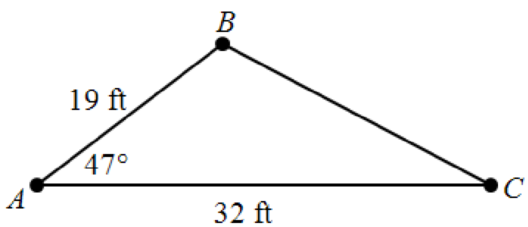
	Draw the triangle.

$\text{Area} = \frac{1}{2}bc(\sin A)$ $\text{Area} = \frac{1}{2}(13)(7)(\sin 43°)$ $\text{Area} = \frac{91}{2}(\sin 43°) \approx 31.031$	Use the formula for the area of a triangle. Substitute in known values and simplify.

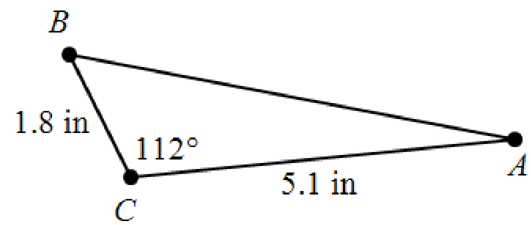
Practice Exercises A

Find the area of each triangle. Round your answer to the nearest thousandth (3 decimal places).

1.



2.



3. $B = 36°$, $a = 3$ cm, $c = 6$ cm

4. $A = 48°$, $b = 20$ m, $c = 40$ m

5. $B = 33°$, $a = 12$ ft, $c = 5.5$ ft

6. $C = 102°$, $a = 16$ mm, $b = 20$ mm

7. $A = 76°$, $b = 11$ m, $c = 24$ m

8. $B = 101°$, $a = 10$ cm, $c = 22$ cm

9. A surveyor wants to mark off a triangular parcel with an area of 0.5 acres (1 acre is equivalent to 43,560 ft^2). One side of the triangle extends 220 feet along a straight road. A second side extends at an angle of $75°$ from one end of the first side. How long should the second side be?

Unit 3 Cluster 1 (G.SRT.10 and G.SRT.11): The Law of Sines and Law of Cosines

Cluster 1: Apply trigonometry to general triangles

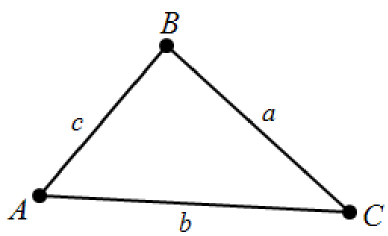
3.1 Prove the Law of Sines.

3.1 Prove the Law of Cosines.

3.1 Use the Law of Sines and the Law of Cosines to solve problems.

Law of Sines

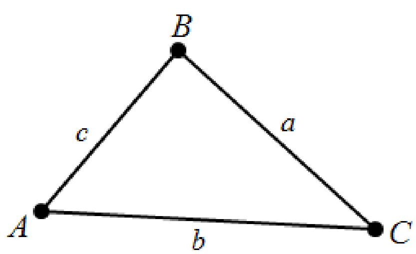
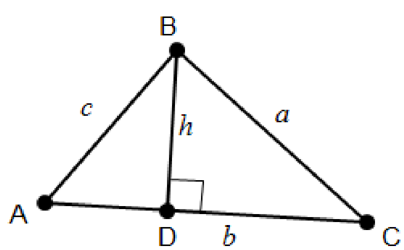
For any $\triangle ABC$, the Law of Sines relates the sine of each angle to the length of the side opposite the angle.



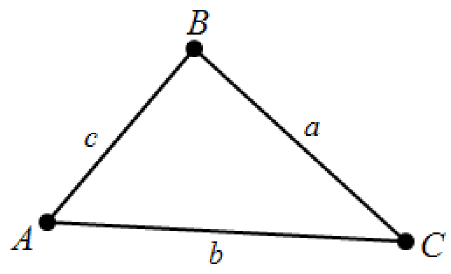
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

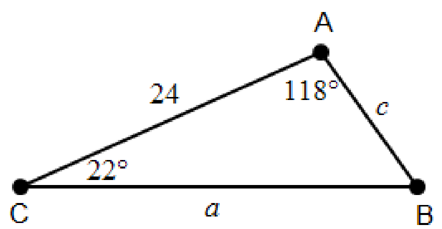
Proof:

	Start with any triangle that has angles A, B, and C and side lengths, *a*, *b*, and *c*, where *a* is the side opposite angle A, *b* is the side opposite angle B, and *c* is the side opposite angle C.
	Construct an altitude, *h*, from one of the angles to the side opposite the angle. The two triangles formed, $\triangle ABD$ and $\triangle BDC$, are right triangles.
$\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$	Use the definition of sine to relate the base angles, $\angle A$ and $\angle C$, to the hypotenuse of each and the altitude.
$c \sin A = h$ and $a \sin C = h$	Use the multiplication property of equality to solve each equation for *h*.
$c \sin A = a \sin C$	Use the transitive property of equality to set the equations equal to one another.
$\frac{\sin A}{a} = \frac{\sin C}{c}$	Use the division property of equality.

You can also derive the Law of Sines from the formula for the area of a triangle given two sides and the included angle.

	All three of these formulas will give the same area for the triangle. $\text{area} = \frac{1}{2}bc(\sin A)$ $\text{area} = \frac{1}{2}ac(\sin B)$ $\text{area} = \frac{1}{2}ab(\sin C)$
$\frac{1}{2}bc(\sin A) = \frac{1}{2}ac(\sin B)$	Set the right side of the first two equations equal to one another.
$bc(\sin A) = ac(\sin B)$ $b(\sin A) = a(\sin B)$	Multiply each side of the equation by 2. Divide each side of the equation by *c*.
$\frac{\sin A}{a} = \frac{\sin B}{b}$	Divide each side by *ab*.

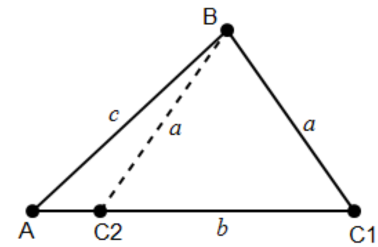
Example 1:
Use the Law of Sines to solve the triangle. Round your answers to three decimal places.



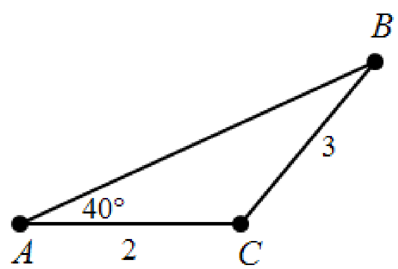
$\frac{a}{\sin A} = \frac{b}{\sin B}$ $\frac{a}{\sin 118°} = \frac{24}{\sin 40°}$	You are given two angles and a side. Use $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ to find the missing side *a*. Substitute in known values. (Hint: $m\angle B = 180° - 118° - 22° = 40°$)
$a = \frac{24}{\sin 40°}(\sin 118°)$ $a \approx 32.967$	Solve for *a*.
$\frac{c}{\sin C} = \frac{b}{\sin B}$ $\frac{c}{\sin 22°} = \frac{24}{\sin 40°}$	Use $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ again to find the missing side *c*.
$c = \frac{24}{\sin 40°}(\sin 22°)$ $c \approx 13.987$	Solve for *c*.
$a \approx 32.967$, $c \approx 13.987$, and $B = 40°$	

The Ambiguous Case (SSA)

If you are given two angles and one side (ASA or AAS), the Law of Sines will easily provide ONE solution for a missing side. However, the Law of Sines has a problem dealing with SSA. If you are given two sides and one angle, where you must find an angle, the Law of Sines could possibly provide you with one or more solutions or even no solution at all.



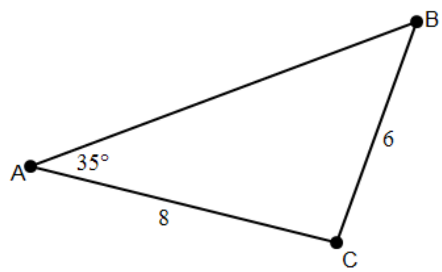
Example 2:
Use the Law of Sines to solve the triangle.

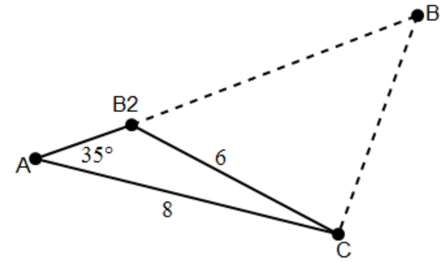


$\frac{\sin 40°}{3}=\frac{\sin B}{2}$ $\frac{2\sin 40°}{3}=\sin B$ $\sin^{-1}\left(\frac{2\sin 40°}{3}\right)=m\angle B$ $m\angle B\approx 25.374°$	Use the Law of Sines to find $m\angle B$.
$180°$ $-\ 25.374°$ $154.626°$ $+\ 40°$ $194.626°$ $194.626°>180°$ There is only one triangle for the given information	Now check to see if more than one triangle exists with the given information. **TEST** 180° - (the found angle) Answer + (given angle) If sum > 180° there is only one triangle If sum < 180° there are two triangles
$m\angle C\approx 180°-(40°+25.375°)$ $m\angle C\approx 114.625°$	Find $m\angle C$.
$\frac{3}{\sin 40°}=\frac{c}{\sin 114.625°}$	Use the Law of Sines to find side *c*.

$c=\dfrac{2\sin 114.625°}{\sin 40°}$ $c \approx 4.243$	
There is only 1 triangle: $c \approx 4.243,\ B \approx 25.374°,$ and $C \approx 114.625°$	

Example 3:
Use the Law of Sines to solve the triangle.
Triangle ABC with sides a = 6, b = 8, and $m\angle A = 35°$.

	Draw the triangle, notice that there are two given sides and one angle.
$\dfrac{\sin 35°}{6}=\dfrac{\sin B}{8}$ $\dfrac{8\sin 35°}{6}=\sin B$ $\sin^{-1}\left(\dfrac{8\sin 35°}{6}\right)=m\angle B$ $m\angle B \approx 49.886°$	Use the Law of Sines to find $m\angle B$.
$180°$ $-\ 49.886°$ $130.114°$ $+\ 35°$ $165.114°$ $165.114° < 180°$ There are two triangles for the given information	Now check to see if more than one triangle exists with the given information. **TEST** 180° - (the found angle) Answer + (given angle) If sum > 180° there is only one triangle If sum < 180° there are two triangles
$m\angle C \approx 180° - (35° + 49.886°)$ $m\angle C \approx 95.114°$	Find $m\angle C$.
$\dfrac{6}{\sin 35°}=\dfrac{c}{\sin 95.114°}$ $c=\dfrac{6\sin 95.114°}{\sin 35°}$ $c \approx 10.419$	Use the Law of Sines to find side c.

The information given can form two different triangles. We draw the second triangle by swinging $\overline{BC}$ towards $\angle A$ forming an isosceles triangle and creating $\triangle AB_2C$.	
$m\angle AB_2C = 180° - m\angle BB_2C$ $m\angle AB_2C = 180° - 49.886°$ $m\angle AB_2C = 130.114°$	Since $\triangle BB_2C$ is an isosceles triangle, $\angle BB_2C \cong \angle B$. $\angle AB_2C$ forms a straight angle with $\angle BB_2C$.
$m\angle ACB_2 = 180° - (35° + 130.114°)$ $m\angle ACB_2 = 14.886°$	Find $\angle ACB_2$.
$\frac{6}{\sin 35°} = \frac{\overline{AB_2}}{\sin 14.886°}$ $\frac{6 \sin 14.886°}{\sin 35°} = \overline{AB_2}$ $\overline{AB_2} \approx 2.687$	Use the Law of Sines to find $\overline{AB_2}$.
Triangle 1: $c \approx 10.419$, $B \approx 49.886°$, and $C \approx 95.114°$ Triangle 2: $c \approx 2.687$, $B \approx 130.114°$, and $C \approx 14.886°$	

Practice Exercises A

Use the Law of Sines to solve each triangle. Round each answer to the nearest thousandth.

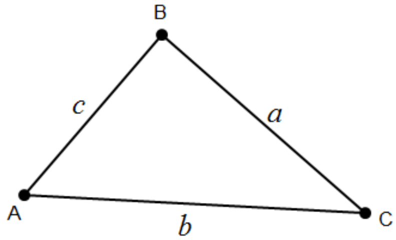
1. $A = 40°,\ B = 30°,\ b = 10$
2. $A = 70°,\ C = 62°,\ a = 7.3$
3. $A = 60°,\ B = 45°,\ b = 3.7$
4. $B = 16°,\ C = 103°,\ c = 12$
5. $B = 62°,\ C = 41°,\ a = 14$
6. $A = 100°,\ C = 35°,\ a = 22$
7. $A = 40°,\ a = 20,\ b = 15$
8. $B = 70°,\ b = 14,\ c = 9$
9. $C = 50°,\ b = 20,\ c = 30$
10. $C = 112°,\ a = 37,\ c = 42.1$
11. $A = 49°,\ a = 32,\ b = 28$
12. $B = 103°,\ b = 61,\ c = 46$

If possible, use the Law of Sines to solve each triangle. There may be one triangle, two triangles, or no triangles at all. Round each answer to the nearest thousandth.

13. $A = 80°,\ a = 17,\ c = 14$
14. $A = 150°,\ a = 9.3,\ b = 41$
15. $C = 36°,\ a = 17,\ c = 16$
16. $B = 60°,\ a = 18,\ b = 16$
17. $C = 30°,\ b = 40,\ c = 10$
18. $B = 30°,\ a = 18,\ b = 9$
19. $C = 38°,\ b = 25,\ c = 21$
20. $B = 38°,\ a = 18,\ b = 12$
21. $A = 63°,\ a = 10,\ c = 8.9$

Law of Cosines

For any $\triangle ABC$, the Law of Cosines relates the length of a side to the other two sides of a triangle and the cosine of the included angle.

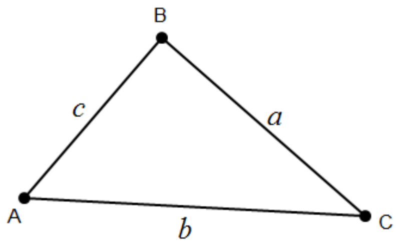
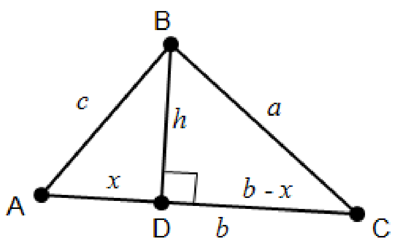


$$a^2 = b^2 + c^2 - 2bc\cos A$$

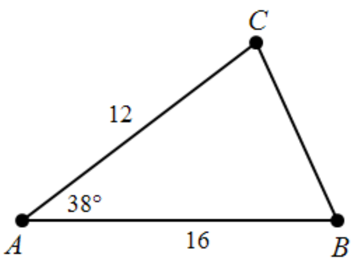
$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Proof:

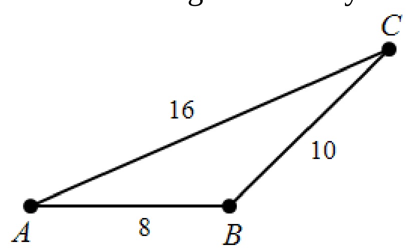
	Start with any triangle that has angles A, B, and C and side lengths, *a*, *b*, and *c*, where *a* is the side opposite angle A, *b* is the side opposite angle B, and *c* is the side opposite angle C.
	Construct an altitude, *h*, from one of the angles to the side opposite the angle. The two triangles formed, $\triangle ABD$ and $\triangle BDC$, are right triangles.
$a^2 = (b-x)^2 + h^2$	Use the Pythagorean theorem to relate the sides of $\triangle BDC$.
$a^2 = b^2 - 2bx + x^2 + h^2$	Expand the binomial.
$a^2 = b^2 - 2b\left[c(\cos A)\right] + c^2$	Use $\cos A = \frac{x}{c}$ or $c(\cos A) = x$ and $x^2 + h^2 = c^2$.
$a^2 = b^2 + c^2 - 2bc(\cos A)$	Rearrange the terms.

Example 4:
Solve the triangle. Round your answers to the nearest thousandth.



$a^2 = 12^2 + 16^2 - 2(12)(16)(\cos 38°)$	Use $a^2 = b^2 + c^2 - 2bc\cos A$. Substitute in known values.
$a^2 = 144 + 256 - 384(\cos 38°)$ $a^2 = 400 - 384(\cos 38°)$ $a^2 \approx 97.404$ $a \approx 9.8693$	Simplify.
$\frac{\sin A}{a} = \frac{\sin B}{b}$ $\frac{\sin 38°}{9.8693} = \frac{\sin B}{12}$ $\frac{12\sin 38°}{9.8693} = \sin B$ $\sin^{-1}\left(\frac{12\sin 38°}{9.8693}\right) = m\angle B$ $48.467° \approx m\angle B$	Use the Law of Sines to find one of the missing angles. Substitute in known values. Simplify.
$m\angle C \approx 180° - 38° - 48.467°$ $m\angle C \approx 93.533°$	Find the third angle.
$a \approx 9.870,\ B \approx 48.467°$, and $C \approx 93.533°$	

Example 5:
Solve the triangle. Round your answers to the nearest thousandth.



$b^2 = a^2 + c^2 - 2ac\cos B$ $16^2 = 10^2 + 8^2 - 2(10)(8)(\cos B)$	Use the Law of Cosines to find the angle opposite the **longest** side. Angle B is opposite the **longest** side. Substitute in known values.
$256 = 100 + 64 - 160(\cos B)$ $256 = 164 - 160(\cos B)$ $92 = -160(\cos B)$ $-0.575 = \cos B$ $\cos^{-1}(-0.575) = m\angle B$ $125.100 \approx m\angle B$	Simplify.

$\dfrac{\sin 125.1°}{16} = \dfrac{\sin A}{10}$ $\dfrac{10\sin 125.1°}{16} = \sin A$ $\sin^{-1}\left(\dfrac{10\sin 125.1°}{16}\right) = m\angle A$ $30.753° \approx m\angle A$	Use the Law of Sines to solve for either of the two remaining angles.
$m\angle C \approx 180° - 125.1° - 30.753°$ $m\angle C \approx 24.147°$	Find the third angle.
$A \approx 30.753°,\ B \approx 125.100°,$ and $C \approx 24.147°$	

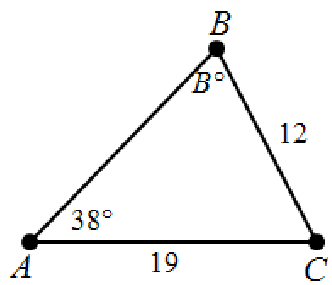
Practice Exercises B

Solve the triangle. Round your answer to the nearest thousandth.

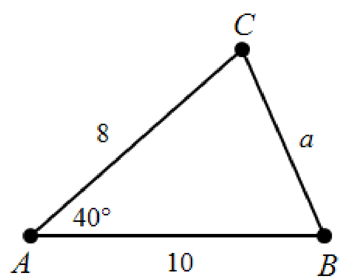
1. $C = 27°,\ a = 5,\ b = 9$
2. $A = 100°,\ b = 4,\ c = 1$
3. $B = 40°,\ a = 80,\ c = 78$
4. $B = 35°,\ a = 43,\ c = 19$
5. $C = 42°,\ a = 5,\ b = 7$
6. $A = 55°,\ b = 12,\ c = 7$
7. $a = 2,\ b = 5,\ c = 4$
8. $a = 10,\ b = 12,\ c = 21$
9. $a = 5,\ b = 7,\ c = 10$
10. $a = 3,\ b = 3,\ c = 5$
11. $a = 11,\ b = 14,\ c = 19$
12. $a = 4,\ b = 7,\ c = 6$

Determine if you would use the Law of Sines or the Law of Cosines to find missing value. Find the missing value. Round your answer to the nearest thousandth.

13.



14.



15. $b = 4,\ c = 8,\ A = 46°$; find a.

16. $a = 10,\ c = 8.9,\ A = 63°$; find b.

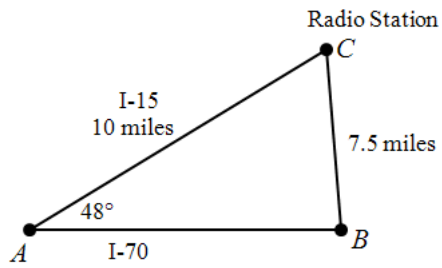
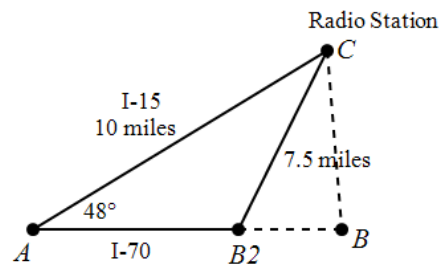
17. $a = 4,\ A = 80°,\ C = 15°$; find c.

18. $a = 12,\ b = 21,\ C = 95°$; find c.

19. $a = 18\ \ b = 17,\ c = 12$; find $m\angle C$.

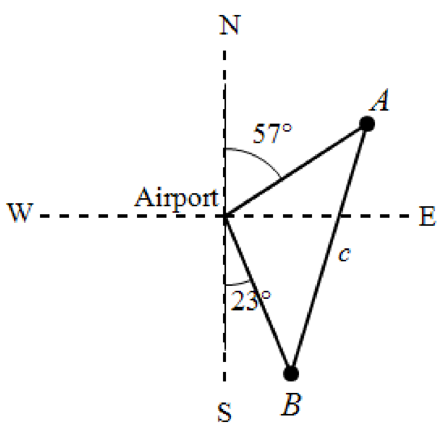
20. $a = 14, B = 41°,\ C = 62°$; find b.

Example 6:
A radio station located adjacent to I-15 is 10 miles from where it connects to I-70. The angle between the two interstates is $48°$. The station can broadcast for a range of 7.5 miles. If Bryce is driving on I-70, between what two distances from the intersection of the two highways can he receive the radio signal?

	Draw a picture to represent the situation. You need to find the length of $\overline{AB}$ in order to find when Bryce can hear the radio station.
$\frac{\sin 48°}{7.5}=\frac{\sin B}{10}$ $\frac{10\sin 48°}{7.5}=\sin B$ $\sin^{-1}\left(\frac{10\sin 48°}{7.5}\right)=m\angle B$ $82.2474° \approx m\angle B$	Use the Law of Sines to solve for angle B.
$m\angle C=180°-(48°+82.2474°)=49.7526°$	Find the measure of angle C.
$\frac{7.5}{\sin 48°}=\frac{AB}{\sin 49.7526°}$ $\frac{7.5\sin 49.7526°}{\sin 48°}=AB$ $7.703 \approx AB$	Use the Law of Sines to find the length of $\overline{AB}$.
$180°$ $-\ 82.2474°$ $97.7526°$ $+\ 48°$ $145.7526°$ $145.7526° < 180°$ There are two triangles for the given information	Now check to see if more than one triangle exists with the given information. **TEST** $180°$ - (the found angle) Answer + (given angle) If sum > $180°$ there is only one triangle If sum < $180°$ there are two triangles
	Two triangles can be formed. Bryce will be able to hear the radio station when he is between $B2$ and B.

$m\angle AB_2C = 180° - m\angle BB_2C$ $m\angle AB_2C = 180° - 82.2474°$ $m\angle AB_2C = 97.7526°$	Since $\triangle BB_2C$ is an isosceles triangle, $\angle BB_2C \cong \angle B$. $\angle AB_2C$ forms a straight angle with $\angle BB_2C$.
$m\angle C = 180° - (48° + 97.7526°) = 34.2474°$	Find the measure of angle C in the new triangle.
$\frac{7.5}{\sin 48°} = \frac{AB_2}{\sin 34.2474°}$ $\frac{7.5 \sin 34.2474°}{\sin 48°} = AB_2$ $5.67959 \approx AB_2$	Use the Law of Sines to find the length of $\overline{AB_2}$.
Bryce can hear the radio station when he is 5.7 to 7.7 miles from where the two interstates intersect on I-70.	

Example 7:
Two airplanes leave an airport at the same time on different runways. One flies on a bearing of N57°E (57° east of north) at 320 miles per hour. The other airplane flies on a bearing of S23°E (23° east of south) at 310 miles per hour. How far apart will the airplanes be after 1.5 hours?

	Draw a triangle to represent the situation. The angle between the two airplanes is $m\angle C = 180° - (57° + 23°) = 100°$. The distance the first plane has traveled after 1.5 hours is $b = 320(1.5) = 480$ miles. The distance the second plane has traveled after 1.5 hours is $a = 310(1.5) = 465$ miles.
$c^2 = 465^2 + 480^2 - 2(465)(480)\cos 100°$ $c^2 = 216,225 + 230,400 - 446,400 \cos 100°$ $c^2 = 446,625 - 446,400 \cos 100°$ $c = \sqrt{446,625 - 446,400 \cos 100°}$ $c \approx 723.976$	Use the Law of Cosines to find the measure of side *c*, which is the distance between the two planes.
The planes are about 724 miles apart after 1.5 hours.	

Practice Exercises C

1. Lighthouse B is 8 miles east of lighthouse A. A boat leaves A and sails 6 miles. At this time, it is sighted from B. If the bearing of the boat from lighthouse B is $S71°W$, how far from lighthouse B is the boat? Round your answer to the nearest mile.

2. You and a friend hike 1.3 kilometers due west from a campsite. At the same time, two other friends hike 1.7 kilometers at a heading of $N17°W$ from the campsite. To the nearest tenth of a kilometer, how far apart are the two groups?

3. An air traffic controller is tracking a plane 2.3 miles due north of the radar tower. A second plane is located 3.6 miles from the tower at a heading of $S72°W$. To the nearest tenth of a mile, how far apart are the two planes?

4. Two observers are 450 feet apart on opposite sides of a flagpole. The angles of elevation from the observers to the top of the pole are $23°$ and $25°$. Find the height of the flagpole to the nearest foot.

5. Two fire-lookout stations are 15 miles apart, with station B directly west of station A. Both stations spot a fire. The bearing of the fire from station A is $S28°W$ and the bearing of the fire from station B is $S49°E$. How far, to the nearest tenth of a mile, is the fire from each lookout station?

6. The player waiting to receive a kickoff stands at the 7 yard line (point A) as the ball is being kicked 61 yards up the field from the opponent's 32 yard line. The kicked ball travels 69 yards at an angle of 10 degrees to the right of the receiver (point B). Find the distance the receiver runs to catch the ball.

7. The dimensions of a triangular flag are 15 inches by 24 inches by 29 inches. To the nearest tenth, what is the measure of the angle formed by the two shorter sides?

8. A leaning wall is inclined at $4°$ from vertical. At a distance of 25 feet from the wall, the angle of elevation to the top is $34°$. Find the height of the wall to the nearest tenth of a foot.

9. A 25-ft water slide has a 10.8-ft. ladder which meets the slide at a $100°$ angle. To the nearest tenth, what is the distance between the end of the slide and the bottom of the ladder?

10. Two observers are 2.4 miles apart on opposite sides of a hot-air balloon. The angle of elevation from observer A is $30°$ and the angle of elevation from observer B is $35°$. Find the altitude of the balloon to the nearest tenth of a mile.

11. After a wind storm, you notice that your 12-foot flagpole may be leaning, but you are not sure. From a point on the ground 10 feet from the base of the flagpole, you find that the angle of elevation to the top is $52°$. Find the angle, to the nearest degree, that the flagpole makes with the ground and determine if it is leaning or not.

12. Two airplanes flying together in formation take off in different directions. One flies due east at 340 mph, and the other flies $N12°E$ at 360 mph. To the nearest tenth, how far apart are the two airplanes 1 hour after they separate, assuming that they fly at the same altitude?

13. One side of a ravine is 18 feet long. The other side is 13 feet long. A 24 foot zipline runs from the top of one side of the ravine to the other. To the nearest tenth, at what angle do the sides of the ravine meet?

14. Two ships leave a harbor at the same time. One ship travels on a bearing of $N\,14^\circ E$ at 12 miles per hour. The other ship travels on a bearing of $S\ 74^\circ W$ at 9 miles per hour. To the nearest tenth of a mile, how far apart will the ships be after three hours?

15. A surveyor needs to determine the distance between two points that lie on opposite banks of a river. Two points, A and C, along one bank are 250 yards apart. The point B is on the opposite bank. Angle A is 64° and angle C is 51°. Find the distance between A and B to the nearest tenth of a yard.

16. The FCC is attempting to locate an illegal radio station. It sets up two monitoring stations, A and B, with station B 30 miles east of station A. Station A measures the illegal signal from the radio station as coming from a direction of 42° east of north. Station B measures the signal as coming from a point 40° west of north. How far is the illegal radio station from monitoring stations A and B?

Unit 3 Cluster 2 (F.TF.1, F.TF.2, and F.TF.3): The Unit Circle

Cluster 2: Extending the domain of trigonometric functions using the unit circle

3.2 Understand radian measure as the length of the arc on the unit circle subtended by the angle.

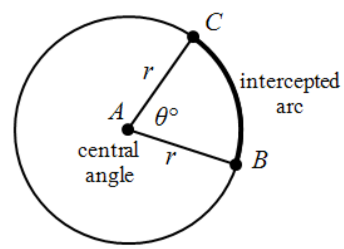
3.2 Use special triangles to determine geometrically the values of sine, cosine, and tangent for $\frac{\pi}{3}, \frac{\pi}{4}$, and $\frac{\pi}{6}$.

3.2 Use the unit circle to express values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$.

VOCABULARY

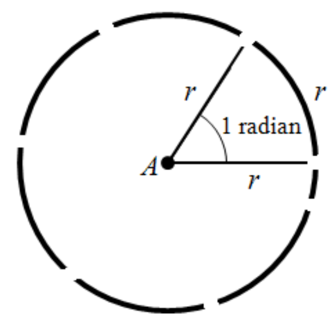
An angle with its vertex at the center of the circle is called a **central angle**.

An **intercepted arc** is the portion of a circle with endpoints on the sides of the central angle and remaining points within the interior of the angle.



A **radian** is the measure of the central angle that intercepts an arc with length equal to the radius of the circle. You can see that it takes 6 radians and a little more (about 0.28) to complete the entire circle. Mathematically, $\frac{C}{r} = \frac{2\pi r}{r} = 2\pi$. Therefore, there are about 6.28 radii around a circle or exactly 2π radians.

A radian, much like an angle in degrees, measures the amount of rotation from the initial side to the terminal side of an angle in terms of the radius.



Recall that the length of the arc intercepted by an angle is proportional to the radius. The formula for the length of an intercepted arc is $l = \frac{\pi\theta}{180°}r$, where θ is measured in degrees and $\frac{\pi}{180°}$ is the constant of proportionality. The radian measure of the angle of the intercepted arc was defined to be the constant of proportionality. Using this fact we can find the measure of the angle whose intercepted arc is *r* and prove that it is 1 radian.

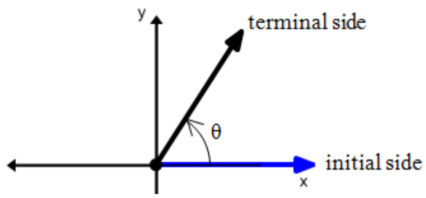
$l = \frac{\pi\theta}{180°}r$	The formula for the length of an intercepted arc.
$r = \frac{\pi\theta}{180°}r$	The length of the intercepted arc is equal to *r*.
$1 = \frac{\pi\theta}{180°}$	Divide each side by *r*.

$1=\dfrac{\pi}{180^\circ}\theta$ $1=\theta$ radian	The constant of proportionality $\dfrac{\pi}{180^\circ}$ is defined to be a radian measure.
The measure of the angle whose intercepted arc is equal to the length of the radius of the circle is 1 radian.	

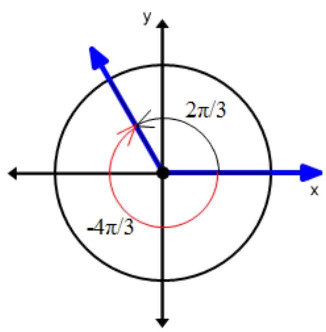
Because all circles are similar, an angle measuring 1 radian will always subtend (intercept) an arc that is equal to 1 radius of the circle.

VOCABULARY

An angle is in **standard position** when the vertex is at the origin and the initial side is on the positive *x*-axis.



Coterminal angles are angles with the same initial and terminal sides, but different measures.

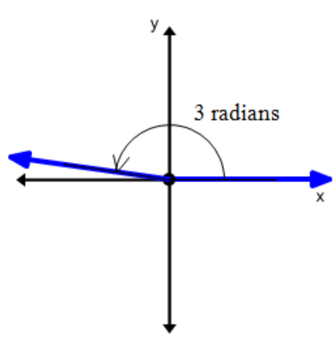
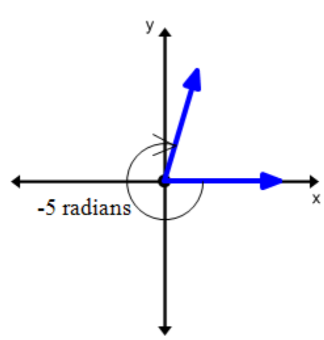


For example $\dfrac{\pi}{6}$ and $-\dfrac{11\pi}{6}$ are coterminal angles, as well as $\dfrac{\pi}{2}$ and $\dfrac{5\pi}{2}$.

Example 1: Sketch each angle in standard position.

a. 3 radians

b. -5 radians

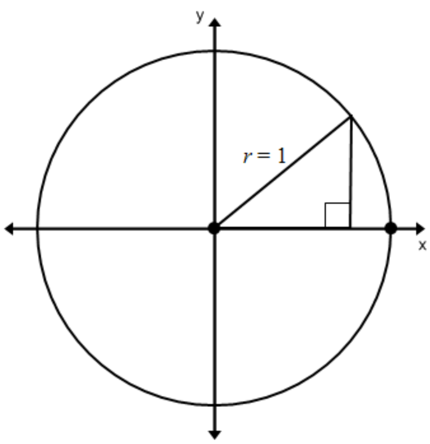
a.	b.
	

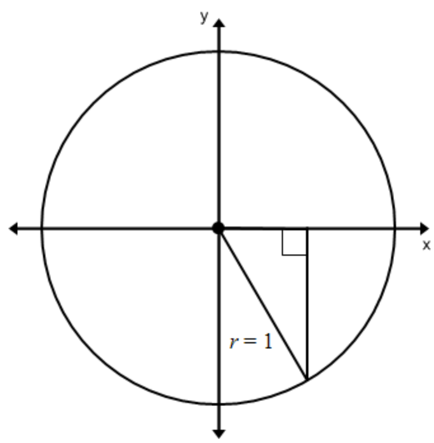
Practice Exercises A

Sketch each angle in standard position.

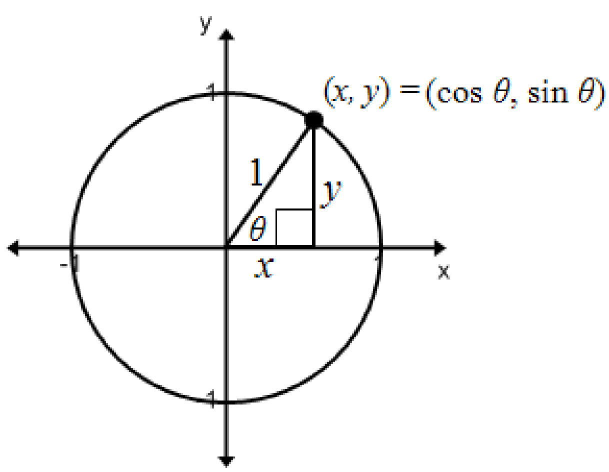
1. 2 radians
2. 7 radians
3. 4.5 radians
4. -4 radians
5. -10 radians
6. -5.5 radians

A circle with radius 1 is called a unit circle. The unit circle provides a connection between trigonometric ratios and the trigonometric functions. We can place it on a coordinate plane and use right triangle trigonometry to find the basic trigonometric ratios. Any right triangle with hypotenuse of length 1 can be drawn in any quadrant of the unit circle.





Trigonometric Ratios for a Circle of Radius 1



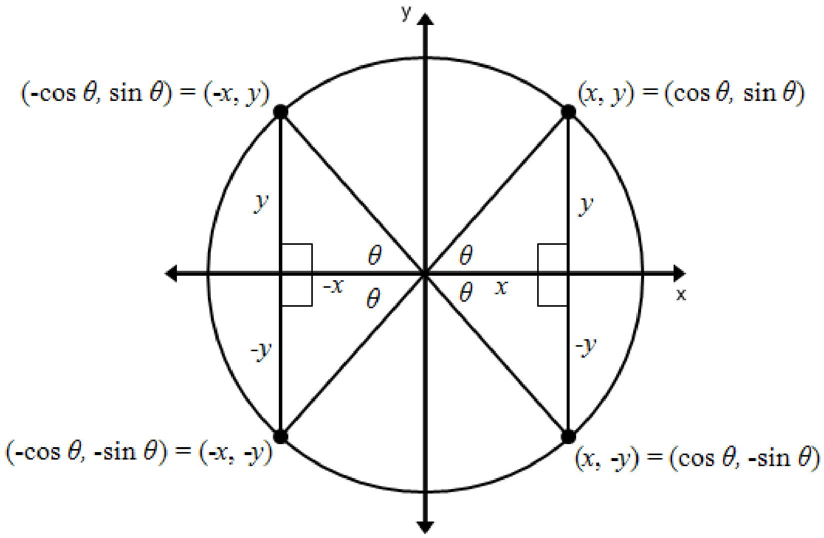
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sin\theta = \frac{y}{1} \qquad \cos\theta = \frac{x}{1}$$

$$\sin\theta = y \qquad \cos\theta = x$$

For any point (x, y) on a unit circle, the *x*-coordinate is the cosine of the angle and the *y*-coordinate is the sine of the angle. Recall that if you reflect any point (x, y) on the coordinate plane across the *x* axis, *y* axis or through the origin, then the following relationships exist:

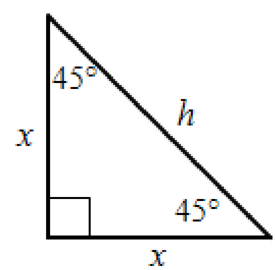
Reflection across the *y* axis $(x, y) \rightarrow (-x, y)$
Reflection across the *x* axis $(x, y) \rightarrow (x, -y)$
Reflection through the origin $(x, y) \rightarrow (-x, -y)$



There are special right triangles with special relationships between the lengths of their sides. These relationships can be used to simplify calculations when finding missing angles and sides.

45°-45°-90° Triangle

The Pythagorean Theorem allows us to derive the relationships that exist for these triangles. Consider a right isosceles triangle with leg lengths *x* and hypotenuse *h*. Since this is a right isosceles triangle then the measures of the angles are $45° - 45° - 90°$

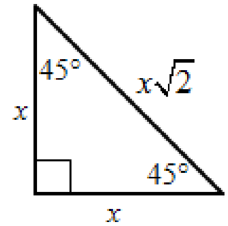


Using the Pythagorean Theorem, we know that $x^2 + x^2 = h^2$. Solving the equation for *x*, you can find the length of the legs.

$$h^2 = x^2 + x^2$$
$$h^2 = 2x^2$$
$$h = \sqrt{2x^2}$$
$$h = x\sqrt{2}$$

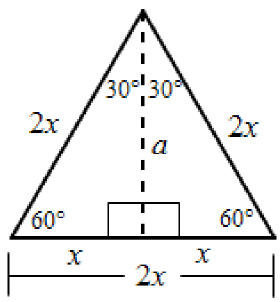
$45° - 45° - 90°$ Triangle

In any $45° - 45° - 90°$ triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of its leg.



$30°$-$60°$-$90°$ Triangles

There is also a special relationship for triangles with angles of $30° - 60° - 90°$. When an altitude, *a*, is drawn from the vertex of an equilateral triangle it bisects the base of the triangle. Two congruent $30° - 60° - 90°$ triangles are formed. If each triangle has a base of length x, then the entire length of the base of the equilateral triangle is $2x$.



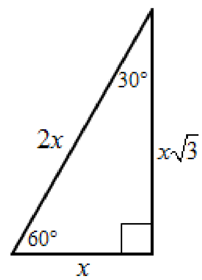
Using one of the right triangles and the Pythagorean Theorem to find the length of the altitude, *a*, we get $a^2 + x^2 = (2x)^2$. Solving it for *a* we get:

$$a^2 + x^2 = (2x)^2$$
$$a^2 = 4x^2 - x^2$$
$$a^2 = 3x^2$$
$$a = \sqrt{3x^2}$$
$$a = x\sqrt{3}$$

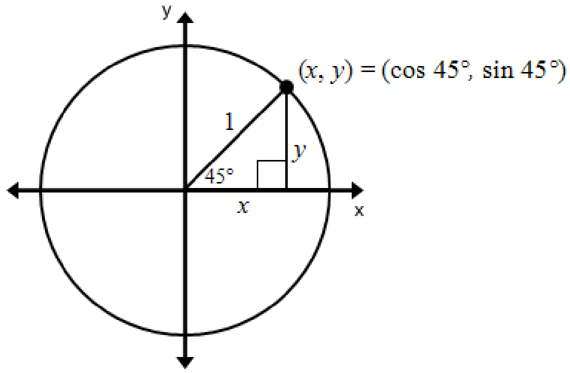
Therefore, in a $30° - 60° - 90°$ triangle, the measures of the side lengths are x, $x\sqrt{3}$, and $2x$.

$30° - 60° - 90°$ Triangles

In any $30° - 60° - 90°$ triangle, the length of the hypotenuse is twice the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.



The figure illustrates how these triangles can be used to derive parts of the unit circle.



Create a right triangle with central angle $45°$. The hypotenuse is length 1 and the legs are lengths *x* and *y*. The angles of the triangle measure $45°, 45°,$ and $90°$. Since two of the angles are congruent this is an isosceles triangle making the lengths of the legs the same, $x = y$. Finding the value of *x* will help us identify the numerical coordinates of the point (x, y).

By the Pythagorean Theorem we know that $x^2 + x^2 = 1^2$. Solving the equation for *x* will give us the value of each leg.

$x^2 + x^2 = 1$	Pythagorean Theorem
$2x^2 = 1$ $x^2 = \frac{1}{2}$	Isolate *x*.
$\sqrt{x^2} = \sqrt{\frac{1}{2}}$	Use properties of rational exponents to simplify.
$x = \frac{1}{\sqrt{2}}$ $x = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ $x = \frac{\sqrt{2}}{2}$	Rationalize the denominator.

Since both legs are equal, $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$.

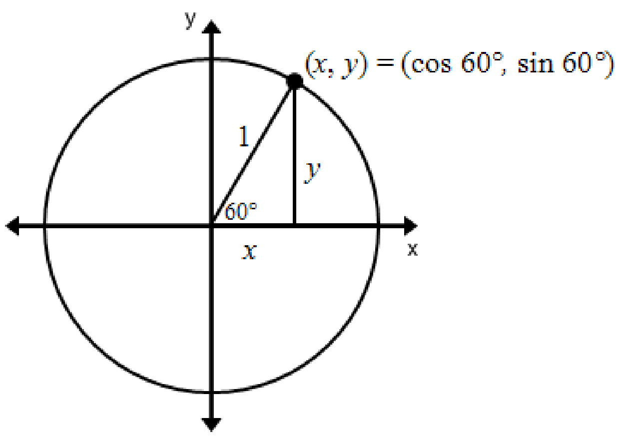
If we relate this to the unit circle where $\theta = 45°$, then the following is true:

$$y = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \sin 45° = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2} \qquad x = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \cos 45° = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

Thus, the point $(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point across the *y*-axis is equivalent to a rotation of $135°$ from the positive *x*-axis. The coordinates of the point are $(-x, y) = (-\cos 45°, \sin 45°) = (\cos 135°, \sin 135°) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
- The reflection of the point through the origin is equivalent to a rotation of $225°$ from the positive *x*-axis. The coordinates of the new point are $(-x, -y) = (-\cos 45°, -\sin 45°) = (\cos 225°, \sin 225°) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.
- The reflection of the point across the *x*-axis is equivalent to a rotation of $315°$ from the positive *x*-axis. The coordinates of the new point are $(x, -y) = (\cos 45°, -\sin 45°) = (\cos 315°, \sin 315°) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

To illustrate a $30° - 60° - 90°$ triangle on the unit circle, create a right triangle with central angle $60°$. The hypotenuse is length 1 and the legs are lengths *x* and *y*.



The length of the hypotenuse in a $30° - 60° - 90°$ is twice the length of the shorter side. Thus, $1 = 2x$. Upon solving the equation for *x* we find that $x = \frac{1}{2}$. We can use this value of *x* to find the length of the longer leg, *y*. The longer leg is $\sqrt{3}$ times the length of the shorter leg. Therefore, $y = x\sqrt{3}$ and since $x = \frac{1}{2}$ then $y = \left(\frac{1}{2}\right)\sqrt{3}$ or $y = \frac{\sqrt{3}}{2}$.

If we relate this to the unit circle where $\theta = °60$ then the following is true:

$$y = \sin\theta = \sin 60° = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \qquad x = \cos\theta = \cos 60° = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

Thus, the point $(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point across the *y*-axis is equivalent to a rotation of $120°$ from the positive *x*-axis. The coordinates of the point are
  $(-x,y)=\left(-\cos 60°,\sin 60°\right)=\left(\cos 120°,\sin 120°\right)=\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.
- The reflection of the point through the origin is equivalent to a rotation of $240°$ from the positive *x*-axis. The coordinates of the new point are
  $(-x,-y)=\left(-\cos 60°,-\sin 60°\right)=\left(\cos 240°,\sin 240°\right)=\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.
- The reflection of the point across the *x*-axis is equivalent to a rotation of $300°$ from the positive *x*-axis. The coordinates of the new point are
  $(x,-y)=\left(\cos 60°,-\sin 60°\right)=\left(\cos 300°,\sin 300°\right)=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.

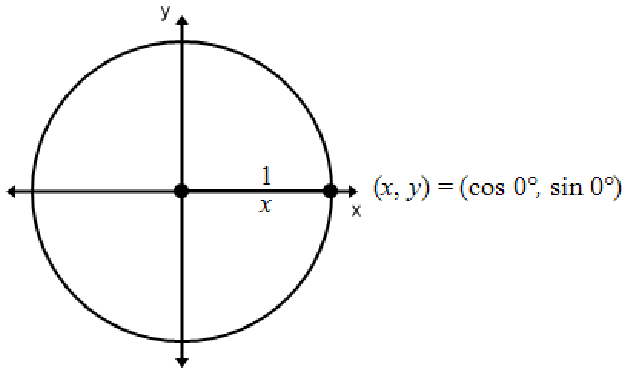
We can also consider the case where $\theta=30°$ and use the $30°-60°-90°$ triangle to find the values of *x* and *y* for this value of θ.

$$y=\sin\theta=\sin 30°=\frac{\frac{1}{2}}{1}=\frac{1}{2} \qquad x=\cos\theta=\cos 30°=\frac{\frac{\sqrt{3}}{2}}{1}=\frac{\sqrt{3}}{2}$$

Thus, the point $(x,y)=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point across the *y*-axis is equivalent to a rotation of $150°$ from the positive *x*-axis. The coordinates of the point are
  $(-x,y)=\left(-\cos 30°,\sin 30°\right)=\left(\cos 150°,\sin 150°\right)=\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$.
- The reflection of the point through the origin is equivalent to a rotation of $210°$ from the positive *x*-axis. The coordinates of the new point are
  $(-x,-y)=\left(-\cos 30°,-\sin 30°\right)=\left(\cos 210°,\sin 210°\right)=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$.
- The reflection of the point across the *x*-axis is equivalent to a rotation of $330°$ from the positive *x*-axis. The coordinates of the new point are
  $(x,-y)=\left(\cos 30°,-\sin 30°\right)=\left(\cos 330°,\sin 330°\right)=\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$.

If we plot all of the points where $\theta = 30°$, $45°$, and $60°$ and their reflections, then we get most of the unit circle. To obtain the rest of the unit circle we have to examine what happens to a point when $\theta = 0°$.

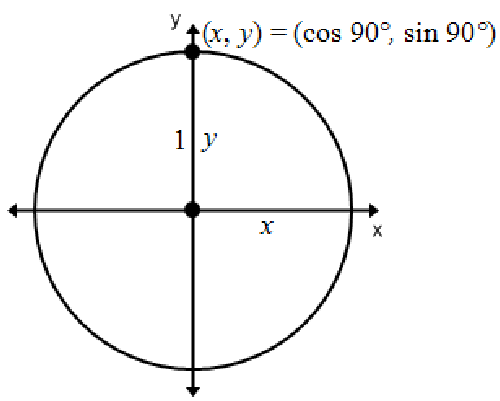


Notice that the value of x is equal to 1 and that y is zero. Thus, when $\theta = 0°$, the point $(x, y) = (1, 0)$. Even though this point does not form a right triangle, any point on a circle can be found by using cosine and sine. Therefore, $\cos 0° = 1$ and $\sin 0° = 0$.

If we reflect the point across the *y*-axis, then the new point is $(-x, y) = (-1, 0)$. This is equivalent to a rotation of $180°$ from the positive *x*-axis. The coordinates of the new point are:

$$(-x, y) = (-\cos 0°, \sin 0°) = (\cos 180°, \sin 180°) = (-1, 0).$$

Finally we need to observe what happens when we rotate a point $90°$ from the positive *x*-axis.

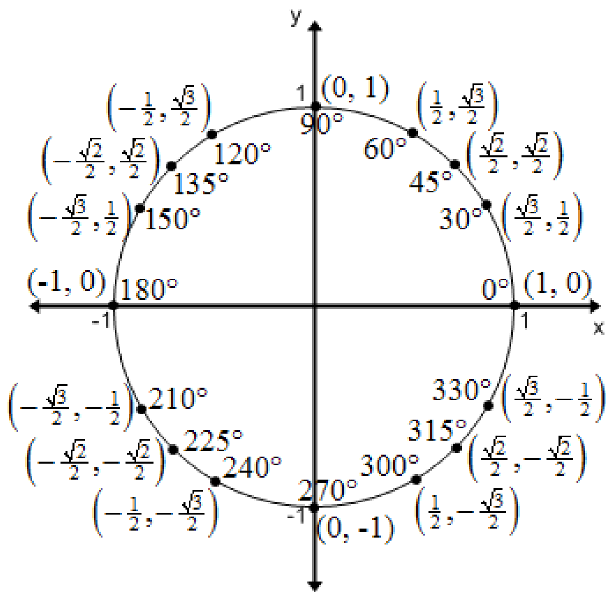


Notice that the value of y is equal to 1 and that x is zero. Thus, when $\theta = 90°$, the point $(x, y) = (0, 1)$. Therefore, $\cos 90° = 0$ and $\sin 90° = 1$.

If we reflect the point across the *x*-axis, then the new point is $(x, -y) = (0, -1)$. This is equivalent to a rotation of $270°$ from the positive *x*-axis. The coordinates of the new point are:

$$(x, -y) = (\cos 90°, -\sin 90°) = (\cos 270°, \sin 270°) = (0, -1).$$

Plotting all of the points, we obtain what is referred to as the unit circle.



The unit circle can be used to find exact values of trigonometric ratios for the angles that relate to the special right triangle angles.

Converting Between Radians and Degrees
To convert degrees to radians, multiply the angle by $\frac{\pi \text{ radians}}{180°}$.
To covert radians to degrees, multiply the angle by $\frac{180°}{\pi \text{ radians}}$.

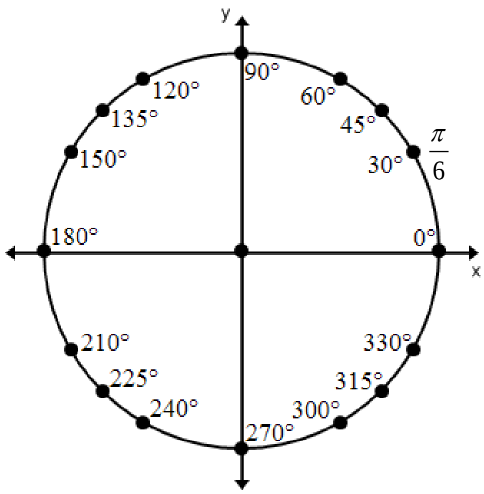
The unit circle can be represented using radian measures instead of degrees.

Example 2:
Convert $30°$ to a radian measure.

$30° \cdot \frac{\pi \text{ radians}}{180°}$ $\pi \text{ radians} \cdot \frac{30°}{180°}$	Multiply the angle by the conversion factor $\frac{\pi \text{ radians}}{180°}$.
$\frac{\pi}{6}$ radians	Simplify the fraction.

Practice Exercises B

Find the radian measure for each angle listed below.



1. $0°$
2. $30° = \frac{\pi}{6}$
3. $45°$
4. $60°$
5. $90°$
6. $120°$
7. $135°$
8. $150°$
9. $180°$
10. $210°$
11. $225°$
12. $240°$
13. $270°$
14. $300°$
15. $315°$
16. $330°$

The unit circle can be used to find exact values of trigonometric ratios for the angles (degree or radian) that relate to the special right triangle angles.

Example 3:
Use the unit circle to find the exact value.

a. $\sin 135°$

b. $\cos \frac{5\pi}{4}$

Answer:

a. The point that has been rotated $135°$ from the positive *x*-axis has coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

The *y*-coordinate is the sine value, therefore, $\sin 135° = \frac{\sqrt{2}}{2}$.

b. The point that has been rotated $\frac{5\pi}{4}$ radians from the positive *x*-axis has coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The *x*-coordinate is the cosine value, therefore, $\cos \frac{5\pi}{4} = -\frac{1}{2}$.

Defining Tangent Values

Another way to write $\tan\theta$ is $\tan\theta = \frac{\sin\theta}{\cos\theta}$. This can be shown algebraically as follows:

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}}$	Use the definition $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta \frac{\text{adjacent}}{\text{hypotenuse}}$.
$\tan\theta = \frac{\text{opposite}}{\text{hypotenuse}} \div \frac{\text{adjacent}}{\text{hypotenuse}}$	Rewrite the division problem so that it is easier to work with.
$\tan\theta = \frac{\text{opposite}}{\text{hypotenuse}} \cdot \frac{\text{hypotenuse}}{\text{adjacent}}$	Dividing by a fraction is the same as multiplying by its reciprocal.
$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \cdot \frac{\text{hypotenuse}}{\text{hypotenuse}}$	Use the commutative property of multiplication to rearrange the terms.
$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \cdot 1$	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}$

Example 4:

Find $\tan\frac{7\pi}{6}$.

Answer:

The coordinates of the point that has been rotated $\frac{7\pi}{6}$ from the positive *x*-axis are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\frac{7\pi}{6} = \frac{\sin\frac{7\pi}{6}}{\cos\frac{7\pi}{6}}$$

Use the coordinates of the point to find $\tan\frac{7\pi}{6}$.

$$\tan\frac{7\pi}{6} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$

$\sin\frac{7\pi}{6} = -\frac{1}{2}$ and $\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$$\tan\frac{7\pi}{6} = -\frac{1}{2} \div -\frac{\sqrt{3}}{2}$$

Rewrite the division problem.

$$\tan\frac{7\pi}{6} = -\frac{1}{2} \cdot -\frac{2}{\sqrt{3}}$$

Dividing by a fraction is the same as multiplying by the reciprocal.

$$\tan\frac{7\pi}{6} = \frac{1}{\sqrt{3}}$$

Simplify.

$$\tan\frac{7\pi}{6} = \frac{\sqrt{3}}{3}$$

Rationalize the denominator.

Tangent Values for the Angles on the Unit Circle

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$
θ	$180°$	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$
θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$

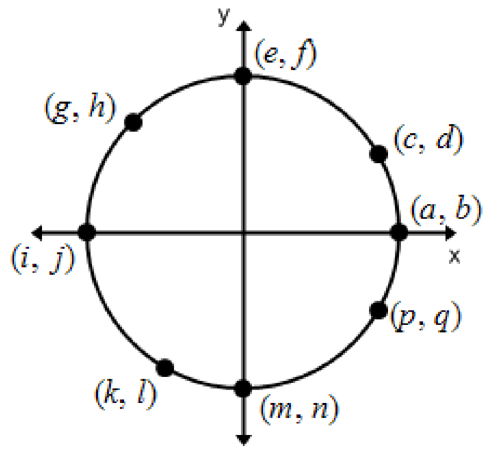
Practice Exercises C

Use the unit circle to find the exact value.

1. $\cos 135°$
2. $\tan 270°$
3. $\sin 300°$
4. $\cos 45°$
5. $\tan 60°$
6. $\sin 120°$
7. $\cos 180°$
8. $\tan 0°$
9. $\sin 210°$
10. $\cos 240°$
11. $\tan 225°$
12. $\sin 315°$
13. $\sin \frac{11\pi}{6}$
14. $\cos \frac{\pi}{6}$
15. $\tan \frac{5\pi}{6}$
16. $\sin \frac{\pi}{2}$
17. $\cos \pi$
18. $\tan \frac{7\pi}{4}$
19. $\sin \frac{5\pi}{4}$
20. $\cos \frac{7\pi}{6}$
21. $\tan \frac{3\pi}{2}$
22. $\sin \frac{\pi}{4}$
23. $\cos \frac{2\pi}{3}$
24. $\tan \frac{\pi}{2}$

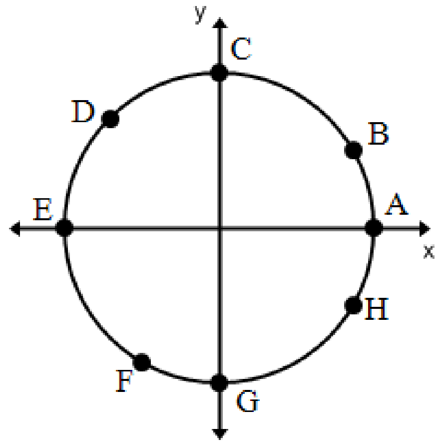
Refer to the diagram. Give the letter that could stand for the function value.

25. $\sin 180°$
26. $\cos 270°$
27. $\cos 30°$
28. $\sin 135°$
29. $\cos 0°$
30. $\sin 330°$
31. $\sin \frac{\pi}{2}$
32. $\cos \frac{4\pi}{3}$
33. $\cos \frac{3\pi}{4}$
34. $\sin \frac{4\pi}{3}$
35. $\cos \frac{11\pi}{6}$
36. $\sin 2\pi$



For the indicated point, tell if the value for $\sin\theta$ or $\cos\theta$ is positive, negative, or neither.

37. $\cos G$	38. $\sin C$
39. $\sin H$	40. $\cos D$
41. $\cos B$	42. $\sin E$
43. $\sin B$	44. $\sin F$
45. $\sin G$	46. $\cos A$
47. $\cos E$	48. $\cos C$



HONORS

Previously we defined the six trigonometric functions. Notice that cosecant, secant, and cotangent are reciprocals of sine, cosine, and tangent, respectively.

The Six Trigonometric Functions

sine $(\theta)=\sin\theta=\dfrac{\text{opposite}}{\text{hypotenuse}}$ **cosecant** $(\theta)=\csc\theta=\dfrac{1}{\sin\theta}=\dfrac{\text{hypotenuse}}{\text{opposite}}$

cosine $(\theta)=\cos\theta=\dfrac{\text{adjacent}}{\text{hypotenuse}}$ **secant** $(\theta)=\sec\theta=\dfrac{1}{\cos\theta}=\dfrac{\text{hypotenuse}}{\text{adjacent}}$

tangent $(\theta)=\tan\theta=\dfrac{\text{opposite}}{\text{adjacent}}$ **cotangent** $(\theta)=\cot\theta=\dfrac{1}{\tan\theta}=\dfrac{\text{adjacent}}{\text{opposite}}$

Example 5:

a. $\sec\dfrac{\pi}{3}$ b. $\cot 330°$

Answer:

a. $\cos\dfrac{\pi}{3}=\dfrac{\text{adjacent}}{\text{hypotenuse}}=\dfrac{1}{2}$ and secant is the reciprocal of cosine therefore,

$\sec\dfrac{\pi}{3}=\dfrac{\text{hypotenuse}}{\text{adjacent}}=\dfrac{2}{1}=2$.

b. $\tan 330°=\dfrac{\sin 330°}{\cos 330°}=-\dfrac{\sqrt{3}}{3}$ and cotangent is the reciprocal of tangent therefore,

$\cot 330°=\dfrac{\cos 330°}{\sin 330°}=-\dfrac{3}{\sqrt{3}}=-\dfrac{3}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{3\sqrt{3}}{3}=-\sqrt{3}$.

Practice Exercises D

Use the unit circle to find the exact value.

1. $\cot\frac{11\pi}{6}$	2. $\csc\frac{\pi}{6}$	3. $\sec\frac{5\pi}{6}$	4. $\cot\frac{\pi}{2}$
5. $\csc\frac{7\pi}{6}$	6. $\sec\frac{7\pi}{4}$	7. $\cot 210^\circ$	8. $\csc 225^\circ$
9. $\sec 240^\circ$	10. $\cot 45^\circ$	11. $\csc 120^\circ$	12. $\sec 60^\circ$

Unit 3 Clusters 2 & 3 (F.TF.2 and F.TF.5): Graphing Sine and Cosine

Cluster 2: Extending the domain of trigonometric functions using the unit circle

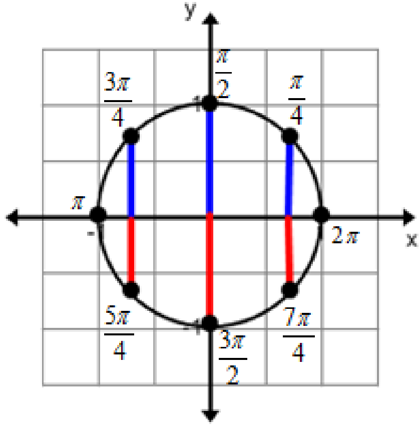
3.2 Explain how the unit circle enables the extension of trigonometric functions to all real numbers.

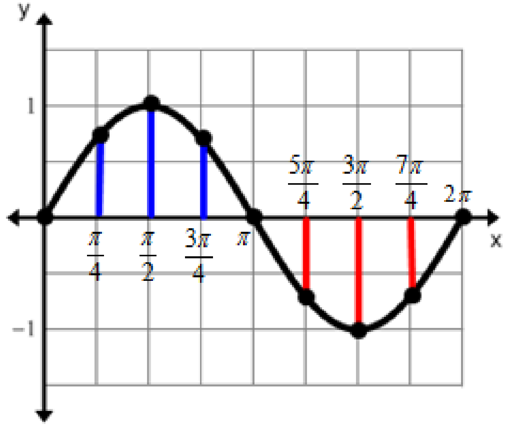
Cluster 3: Model periodic phenomena with trigonometric functions

3.3 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

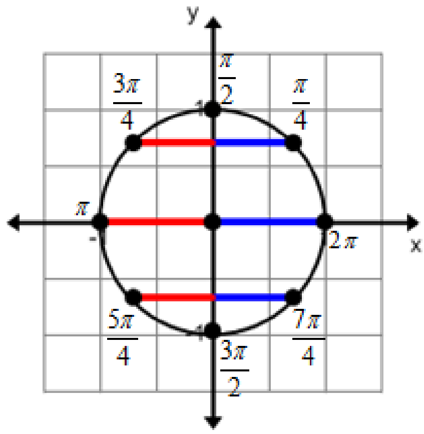
Graphing Sine and Cosine

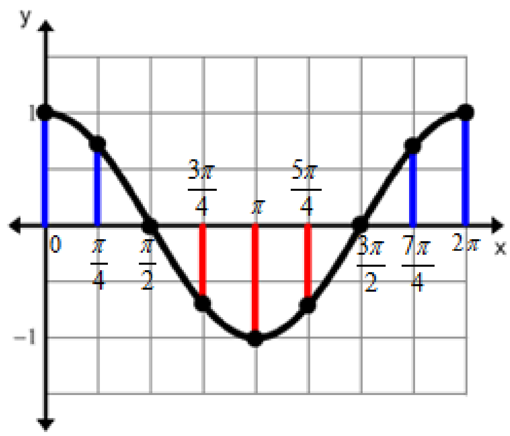
Plotting every angle and its corresponding sine value, which is the *y*-coordinate, for different angles on the unit circle, allows us to create the sine function where $x=\theta$ and $y=\sin\theta$.





Plotting every angle and its corresponding cosine value, which is the *x*-coordinate, for different angles on the unit circle, allows us to create the cosine function where $x=\theta$ and $y=\cos\theta$.





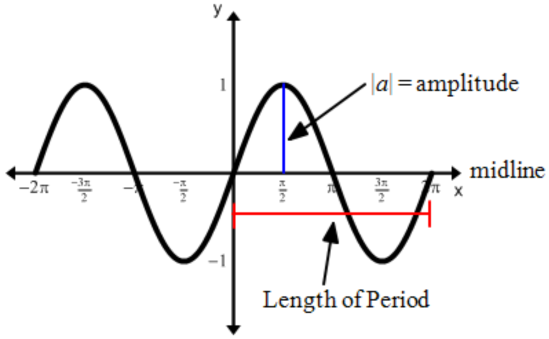
Keep in mind that we can extend this concept to all real numbers because rotating around the circle multiple times maps the new angle on to an existing coterminal angle. This type of function is called a **periodic** function because it has a pattern that repeats.

General Equations and Graphs

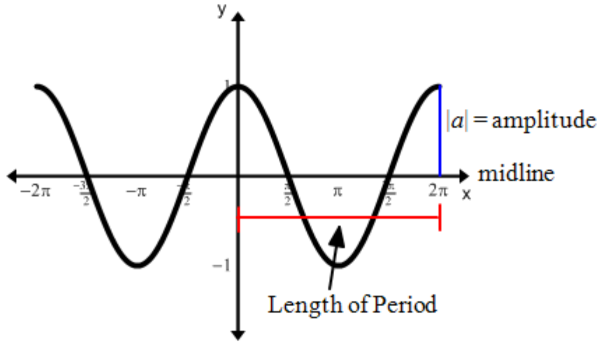
$$f(x) = a\sin(bx) + k \qquad \text{and} \qquad f(x) = a\cos(bx) + k$$

The domain of each function is the set of all real numbers, $(-\infty, \infty)$. The range of each function is $[-1,1]$.

$f(x) = \sin(x)$



$f(x) = \cos(x)$



VOCABULARY

The **amplitude**, $|a|$, is half the difference between the maximum and minimum values of the function.

The **period**, $\frac{2\pi}{|b|}$, is the interval length needed to complete one cycle.

The **frequency**, $\frac{|b|}{2\pi}$, is the number of complete cycles a periodic function makes in a specific interval.

The **midline**, k, is the horizontal line that cuts the trigonometric function in half.

Example 1:
Identify the amplitude and period, then sketch one period of the graph.

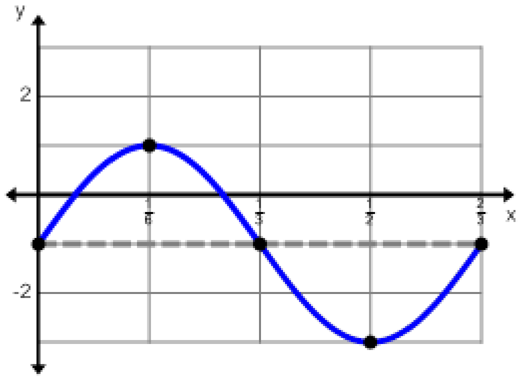
a. $f(x) = 2\sin(3\pi x) - 1$

b. $f(x) = -5\cos\left(\frac{1}{2}x\right) + 3$

a. $f(x) = 2\sin(3\pi x) - 1$

Amplitude $= |2| = 2$

Period $= \dfrac{2\pi}{|3\pi|} = \dfrac{2}{3}$



Draw the midline located at $y = -1$. Label the period $\frac{2}{3}$ and then divide that section into four even segments.

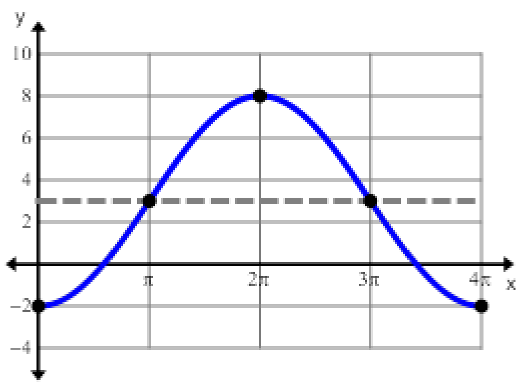
Determine the maximum, $y = k + a = -1 + 2 = 1$, and minimum, $y = k - a = -1 - 2 = -3$, values.

The sine function starts at the midline, rises to the maximum, decreases back to the midline, decreases to the minimum, and finally increases back to the midline. Plot these five points (midline, maximum, midline, minimum, midline) and connect the points.

b. $f(x) = -5\cos\left(\frac{1}{2}x\right) + 3$

Amplitude $= |-5| = 5$

Period $= \dfrac{2\pi}{\left|\frac{1}{2}\right|} = 4\pi$



Draw the midline located at $y = 3$. Label the period 4π and then divide that section into four even segments.

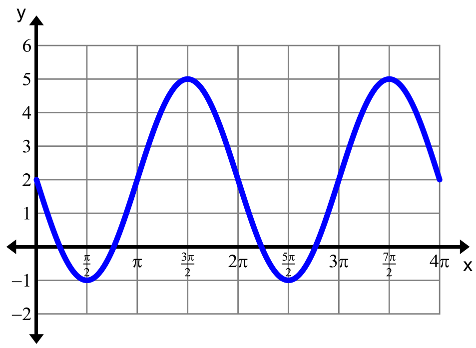
Determine the maximum, $y = k + a = 3 + 5 = 8$, and minimum, $y = k - a = 3 - 5 = -2$, values.

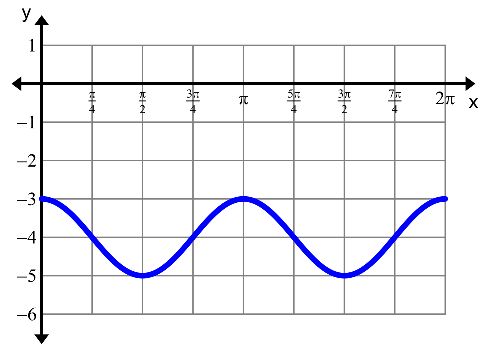
Ordinarily the cosine function starts at its maximum value, decreases to the midline, decreases to the minimum, increases to the midline, and increases to the maximum. This function has been reflected over the *x*-axis so plot the minimum, midline, maximum, midline, and then the minimum again and connect the points.

Example 2:
Identify the amplitude and period, and determine where the midline is located. Then write the equation for the function.

a. Use $f(x)=\sin x$ for your parent graph.

b. Use $f(x)=\cos x$ for your parent graph.





a. $y=2$	Determine the midline.
Amplitude $=\frac{5-(-1)}{2}=3$	Find the difference between the maximum and minimum values and divide by 2. Maximum Value 5 Minimum Value -1
Period $=2\pi$	Notice the graph goes from its minimum to its maximum and back to the midline in the length of 2π.
$f(x)=-3\sin x+2$	$2\pi=\frac{2\pi}{\lvert b\rvert}$ $\lvert b\rvert=\frac{2\pi}{2\pi}$ $b=1$ The sine function usually starts at the midline and goes to its maximum, but this graph starts at the midline and goes to the minimum so the function has been reflected.

b. $y=-4$	Determine the midline
Amplitude $=\frac{-3-(-5)}{2}=1$	Find the difference between the maximum and minimum values and divide by 2. Maximum Value -3 Minimum Value -5
Period $=\pi$	Notice that the graph goes from its minimum to its maximum and back to the midline in the length of π.
$f(x)=\cos 2x-4$	$\pi=\frac{2\pi}{\lvert b\rvert}$ $\lvert b\rvert=\frac{2\pi}{\pi}$ $b=2$

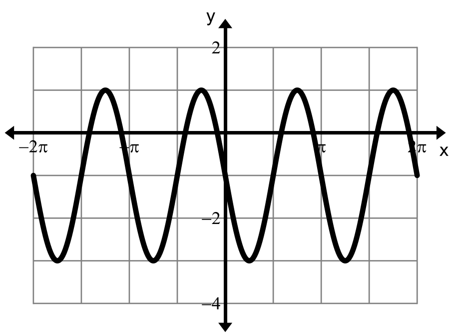
Practice Exercises A

Identify the amplitude and period. Then sketch one period of the graph.

1. $f(x) = \sin x + 2$
2. $f(x) = \sin x - 2$
3. $f(x) = \cos x + 3$
4. $f(x) = \cos x - 3$
5. $f(x) = 2\sin\left(\frac{1}{2}x\right) + 1$
6. $f(x) = 2\cos\left(\frac{1}{2}x\right) - 1$
7. $f(x) = -\sin(\pi x)$
8. $f(x) = -3\cos(2\pi x)$
9. $f(x) = 4\sin(2x) - 2$
10. $f(x) = 4\cos\left(\frac{2}{3}x\right) + 3$
11. $f(x) = -\dfrac{1}{2}\sin x + 3$
12. $f(x) = 2\cos x - 4$

Identify the amplitude and period, and determine where the midline is located. Then write the equation for the function. Use $f(x) = \sin(x)$ for the parent function.

13.



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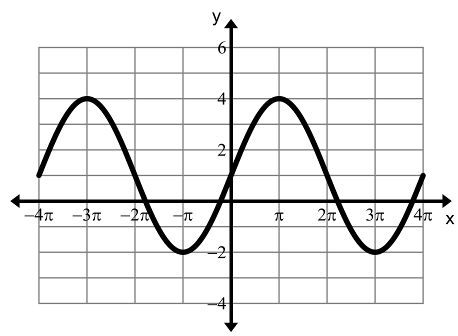

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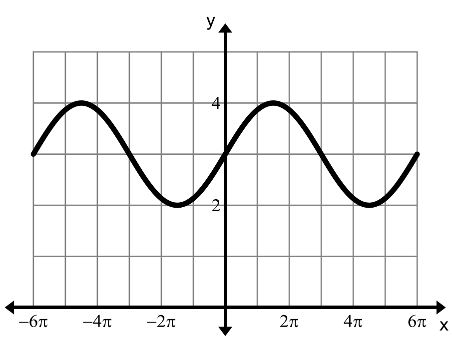

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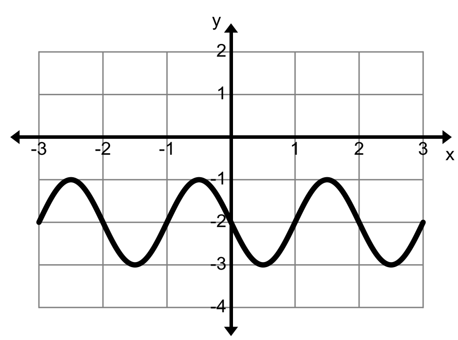
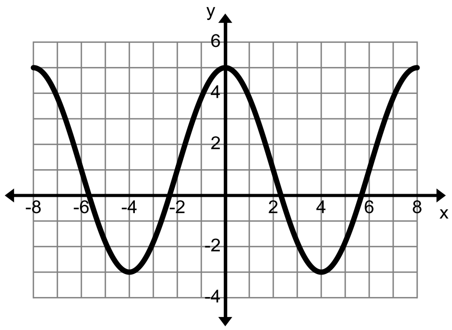


Identify the amplitude and period, and determine where the midline is located. Then write the equation for the function. Use $f(x)=\cos(x)$ for the parent function.

17.



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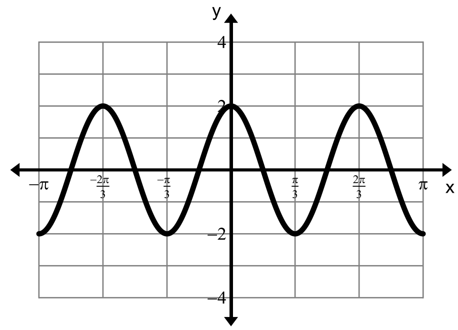

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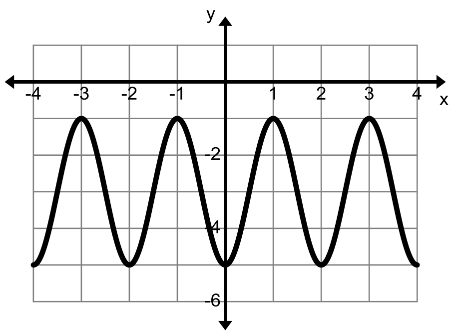

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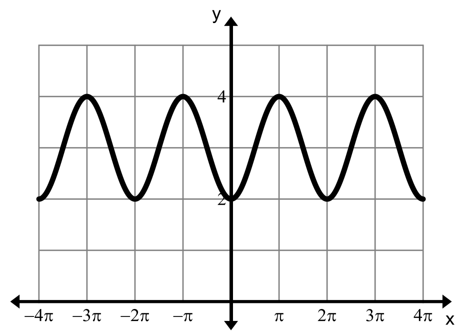
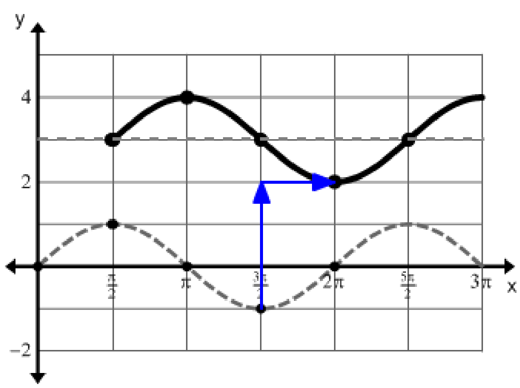


Horizontal Shift (Phase Shift)

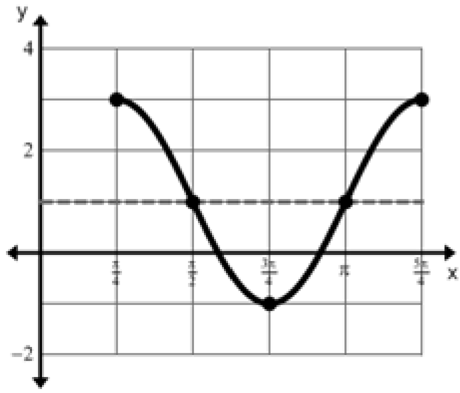
VOCABULARY
When a horizontal shift is performed on a trigonometric function it is called a **phase shift**. The general equations are $f(x) = a\sin\left[b(x-h)\right]+k$ or $f(x) = a\cos\left[b(x-h)\right]+k$, where h is the number of units the graph is shifted horizontally.

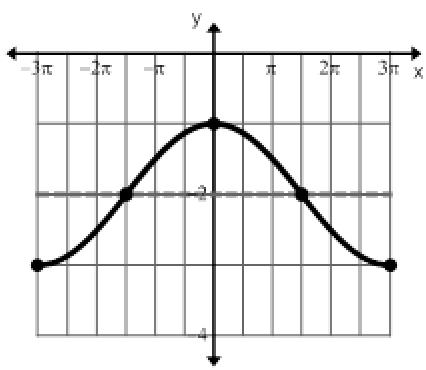
Example 3:
Identify the amplitude, period, phase shift, and vertical shift, then sketch one period of the graph.

a. $f(x) = \sin\left(x - \frac{\pi}{2}\right) + 3$ b. $f(x) = 2\cos\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$ c. $f(x) = -\cos\left(\frac{1}{3}x + \pi\right) - 2$

a. $f(x) = \sin\left(x - \frac{\pi}{2}\right) + 3$ Amplitude: 1 Period: $\frac{2\pi}{\lvert 1 \rvert} = 2\pi$ Phase shift: right $\frac{\pi}{2}$ units Vertical shift: up 3 units	The amplitude is $\lvert a \rvert$. The period is $\frac{2\pi}{\lvert b \rvert}$. $h = \frac{\pi}{2}$ $k = 3$
 	Each point has been shifted up 3 units and to the right $\frac{\pi}{2}$ units. Follow the pattern of midline, max, midline, min, and midline to plot the shifted sine function.

b. $f(x) = 2\cos\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$ Amplitude: 2 Period: $\frac{2\pi}{\lvert 2 \rvert} = \pi$ Phase shift: right $\frac{\pi}{4}$ units Vertical shift: up 1 unit	The amplitude is $\lvert a \rvert$. The period is $\frac{2\pi}{\lvert b \rvert}$. $h = \frac{\pi}{4}$ $k = 1$

	The first point has been shifted $\frac{\pi}{4}$ units to the right and up 1 unit (amplitude of 2). Follow the pattern of max, midline, min, midline, and max to plot the rest of the shifted cosine function.

c. $f(x) = -\cos\left(\frac{1}{3}x + \pi\right) - 2$ $f(x) = -\cos\left[\frac{1}{3}(x + 3\pi)\right] - 2$ Amplitude: 1 Period: $\frac{2\pi}{\left\lvert\frac{1}{3}\right\rvert} = 6\pi$ Phase shift: right 3π units Vertical shift: down 2 units	Rewrite the function so that the coefficient of x has been factored out of both terms. The amplitude is $\lvert a\rvert$. The period is $\frac{2\pi}{\lvert b\rvert}$. $h = 3\pi$ $k = 2$
	The first point has been shifted 3π units to the left and down 2 units. The function has been reflected across the x-axis. Follow the pattern of min, midline, max, midline, and min to plot the rest of the shifted cosine function.

Practice Exercises B

Identify the amplitude, period, phase shift, and vertical shift, then sketch one period of the graph.

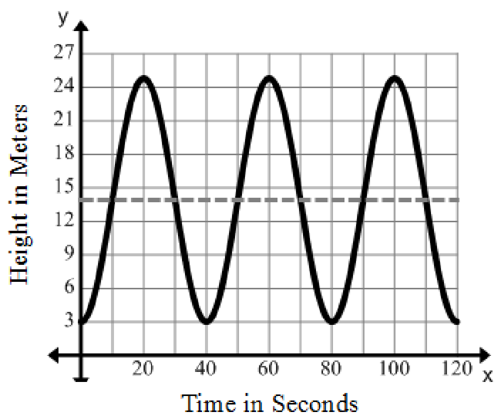
1. $f(x) = \sin\left(x - \frac{\pi}{2}\right) + 2$
2. $f(x) = \cos(x + \pi) - 1$
3. $f(x) = -2\sin(x - \pi)$
4. $f(x) = -\cos\left[\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right]$
5. $f(x) = 3\sin\left[2\left(x - \frac{3\pi}{2}\right)\right]$
6. $f(x) = \cos\left[3\left(x + \frac{3\pi}{2}\right)\right] + 1$
7. $f(x) = \sin\left(3x - \frac{3\pi}{2}\right) + 4$
8. $f(x) = -3\cos(2x + \pi)$
9. $f(x) = \sin\left(\frac{3}{2}x + \frac{3\pi}{4}\right) - 3$

Modeling Periodic Phenomena

Example 4:

The Ferris wheel at Lagoon has a diameter of 21.8 meters. It rotates on a platform that is 3 meters above the ground. The Ferris wheel completes one revolution in 40 seconds. Write an equation to model the situation. Then sketch a graph of height versus time, extending the graph for more than one revolution.

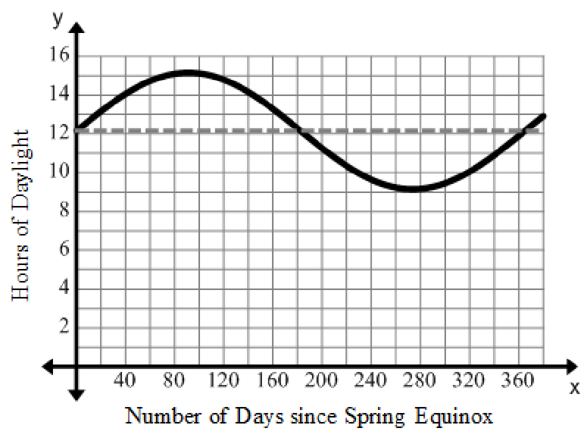
$\lvert a \rvert = \frac{24.8-3}{2} = \frac{21.8}{2} = 10.9$	The amplitude is half the difference of the maximum and minimum values. Minimum: 3 meters Maximum: 3 + 21.8 = 24.8 meters
$40 = \frac{2\pi}{b}$ $b = \frac{2\pi}{40} = \frac{\pi}{20}$	The period is 40 seconds. Find the value of *b*.
$k = 10.9 + 3 = 13.9$	The center of the Ferris wheel is 10.9 meters above the platform. The value of *k* is the distance the center of the Ferris wheel is from the ground.
$f(x) = -10.9\cos\left(\frac{\pi}{20}x\right) + 13.9$	In order to get on the Ferris wheel, the cart must be at its minimum value. Cosine usually starts at its maximum, but if you reflect it across the *x*-axis, then it will start at its minimum.



Example 5:

In Salt Lake City, Utah, at the spring equinox (March 20, 2013) there were 12 hours and 9 minutes of daylight. The longest day (June 20, 2013) and shortest day (December 21, 2013) of the year vary from the equinox by approximately 3 hours. Write a sine function that relates the number of days to the variation of daylight hours in Salt Lake City. Graph the model, showing at least one year.

$\lvert a \rvert = 3$	The amount of daylight varies from the equinox by 3 hours so the amplitude is 3.
$365 = \frac{2\pi}{b}$ $b = \frac{2\pi}{365}$	The period is 365 days. Find the value of *b*.
$k = 12 + \frac{9}{60} = 12.15$	The midline is at 12 hours and 9 minutes because the amount of daylight is ± 3 hours from this value.
$f(x) = 3\sin\left(\frac{2\pi}{365}x\right) + 12.15$	Substitute in known values.



Practice Exercises C

1. A buoy oscillates up and down as waves go past. The buoy moves a total of 3.6 feet from its low point to its high point, and then returns to its high point every 8 seconds. Write a cosine function modeling the buoy's vertical position at any time *t*.

2. A Ferris wheel 50 feet in diameter makes one revolution every 40 seconds. The center of the wheel is 30 feet above the ground. Write a cosine function to model the height of a car on the Ferris wheel at any time *t*.

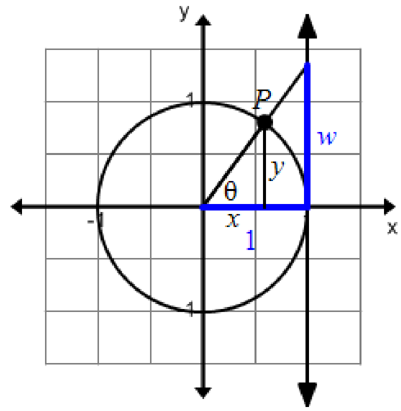
3. Low tide is at 10:15 am and high tide is at 4:15 pm. The water level varies 64 inches between low and high tide. Write a cosine function to represent the change in water level.

4. The lowest pitch a human can easily hear has a frequency of 30 cycles per second. Write a sine function representing the sound wave of the pitch. (Amplitude is 1)

5. The highest pitch a human can easily hear has a frequency of 20,000 cycles per second. Write a sine function representing the sound wave of the pitch. (Amplitude is 1)

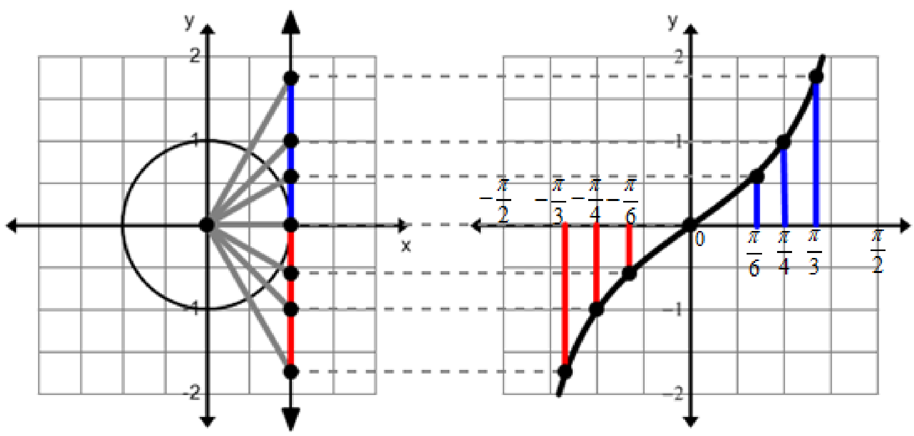
6. In Buenos Aires, Argentina, the average monthly temperature is the highest in January and the lowest in July. It ranges from $76°$F to $51°$F. Write a cosine function that models the change in temperature according to the month of the year.

Graphing Tangent (Honors)



In order to graph the tangent function, $f(x) = \tan x$, we have to examine the relationship between two similar triangles drawn at the right. The smaller triangle is inscribed in the unit circle and intersects the circle at the point $P(x, y)$. The larger triangle shares a vertex with the smaller triangle, but its shorter leg is the length of the radius of the circle. Setting up the proportion we get: $\frac{\text{long leg}}{\text{short leg}} = \frac{y}{x} = \frac{w}{1}$. Simplifying this statement we can see that $w = \frac{y}{x}$. Remember that $y = \sin\theta$, $x = \cos\theta$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$. Thus, the length of the longer leg of the triangle is equal to $w = \frac{y}{x} = \tan\theta$.

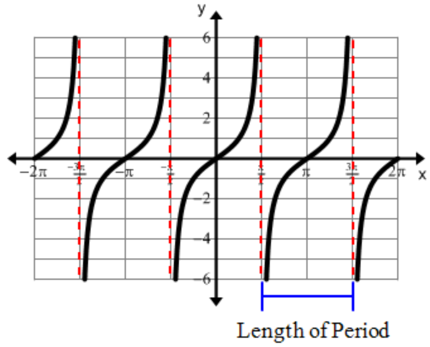
If we draw more similar triangles for different angles on the unit circle, then we would be able to plot the graph of the tangent function where $x = \theta$ and $y = \tan\theta$. Recall that tangent is undefined when $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$. The graph would have vertical asymptotes at those values.



General Equation and Graph

$$f(x) = a\tan(bx) + k$$

The domain of the function is all real numbers except odd multiples of $\frac{\pi}{2}$. The range of each function is the set of all real numbers. The graph crosses the *y*-axis half way between the two asymptotes.



VOCABULARY

Period, $\frac{\pi}{|b|}$, is the interval length needed to complete one cycle.

Frequency, $\frac{|b|}{\pi}$, is the number of complete cycles a periodic function makes in a specific interval.

The **Asymptotes** are determined using the following:

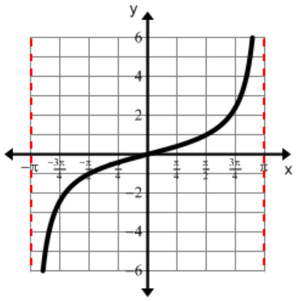
The first set, centered around the origin is given by $x = 0 \pm \frac{\pi}{2|b|}$. To determine the remaining asymptotes, add the period to the previous asymptote.

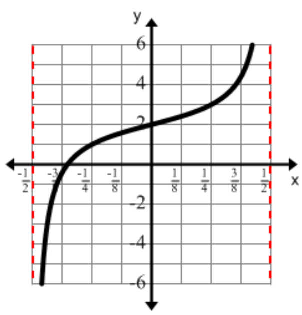
Example 6:
Identify the period, vertical asymptotes, and *y*-intercept, then sketch one period of the graph.

a. $f(x) = \tan\left(\frac{1}{2}x\right)$

b. $f(x) = \tan(\pi x) + 2$

a. Period $= \frac{\pi}{\lvert b\rvert} = \frac{\pi}{\lvert\frac{1}{2}\rvert} = 2\pi$	Draw the asymptotes. The *x*-coordinate of the *y*-intercept is located half way between $\pm\pi$ and the *y*-coordinate is the vertical shift. Sketch the graph.
Asymptotes $x = 0 \pm \frac{\pi}{2\lvert b\rvert}$ $= 0 \pm \frac{\pi}{2\lvert\frac{1}{2}\rvert}$ $= \pm\pi$	

The y-intercept is at $(0,0)$	
	

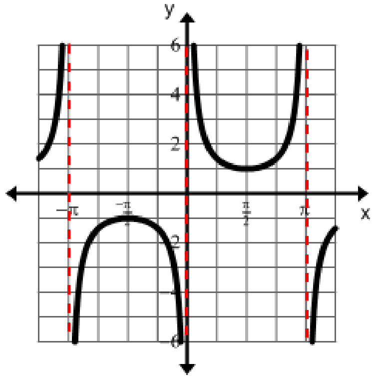
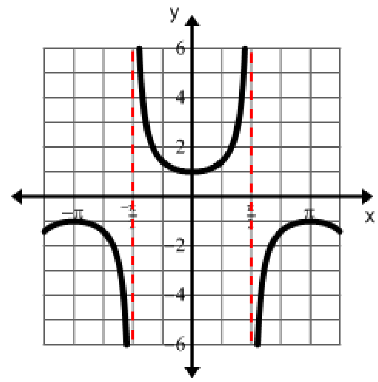
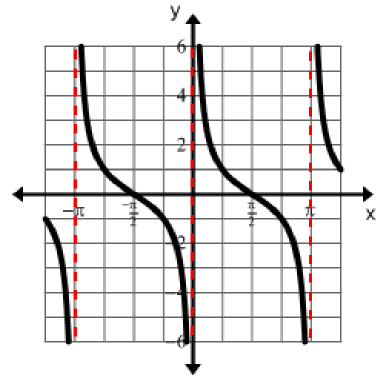
b. Period $=\frac{\pi}{\lvert b\rvert}=\frac{\pi}{\lvert\pi\rvert}=1$	Draw the asymptotes. The x-coordinate of the y-intercept is located half way between $\pm\frac{1}{2}$ and the y-coordinate is the vertical shift. Sketch the graph.
Asymptotes $x=0\pm\frac{\pi}{2\lvert b\rvert}$ $=0\pm\frac{\pi}{2\lvert\pi\rvert}$ $=\pm\frac{1}{2}$	
The y-intercept is at $(0,2)$	
	

Practice Exercises D

Identify the period, vertical asymptotes, and y-intercept, then sketch one period of the graph.

1. $f(x)=\tan\left(\frac{x}{4}\right)+3$
2. $f(x)=\tan(2x)+1$
3. $f(x)=\tan x-4$
4. $f(x)=-\tan\left(\frac{1}{2}x\right)$
5. $f(x)=-2\tan(2x)$
6. $f(x)=5\tan(x)-3$
7. $f(x)=3\tan(2x)-2$
8. $f(x)=-\tan(\pi x)+4$
9. $f(x)=\frac{1}{2}\tan\left(\frac{\pi}{2}x\right)$

Reciprocal Trigonometric Functions (Honors)

Graphs of the Reciprocal Trigonometric Functions		
$f\left(x\right)=\csc x$ 	$f\left(x\right)=\sec x$ 	$f\left(x\right)=\cot x$ 

Summary of the Basic Trigonometric Functions

Function	Period	Domain	Range	Asymptotes	Zeros	Even/Odd
$f\left(x\right)=\sin x$	2π	$\left(-\infty,\infty\right)$	$\left[-1,1\right]$	None	$n\pi$	Odd
$f\left(x\right)=\cos x$	2π	$\left(-\infty,\infty\right)$	$\left[-1,1\right]$	None	$\frac{\pi}{2}+n\pi$	Even
$f\left(x\right)=\tan x$	π	$x\neq\frac{\pi}{2}+n\pi$	$\left(-\infty,\infty\right)$	$x=\frac{\pi}{2}+n\pi$	$n\pi$	Odd
$f\left(x\right)=\csc x$	2π	$x\neq n\pi$	$\left(-\infty,-1\right]\cup\left[1,\infty\right)$	$x=n\pi$	None	Odd
$f\left(x\right)=\sec x$	2π	$x\neq\frac{\pi}{2}+n\pi$	$\left(-\infty,-1\right]\cup\left[1,\infty\right)$	$x=\frac{\pi}{2}+n\pi$	None	Even
$f\left(x\right)=\cot x$	π	$x\neq n\pi$	$\left(-\infty,\infty\right)$	$x=n\pi$	$\frac{\pi}{2}+n\pi$	Odd

Unit 3 Clusters 2 and 3 HONORS (F.TF.4): Symmetry and Periodicity

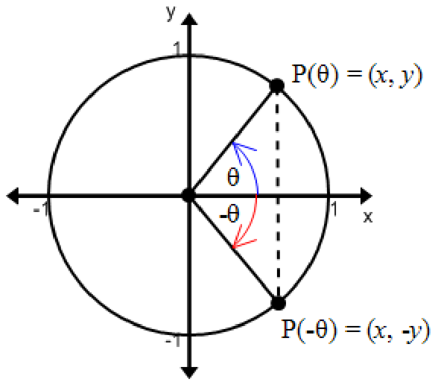
Cluster 2: Extending the domain of trigonometric functions using the unit circle

3.2 Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

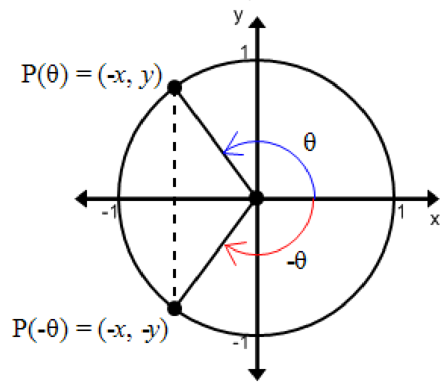
Odd and Even Symmetry

Symmetry in a unit circle shows that for any real number θ, the points $P(\theta)$ and $P(-\theta)$, where $x=\cos\theta$ and $y=\sin\theta$, located on the terminal side of an angle θ will have the same cosine values and opposite sine values. This holds true regardless of which quadrant contains the terminal side of the angle.

1st and 4th Quadrants



2nd and 3rd Quadrants



Therefore, $\cos(-\theta)=\cos(\theta)$ making cosine an even function and $\sin(-\theta)=-\sin(\theta)$ making sine an odd function.

Example 1:

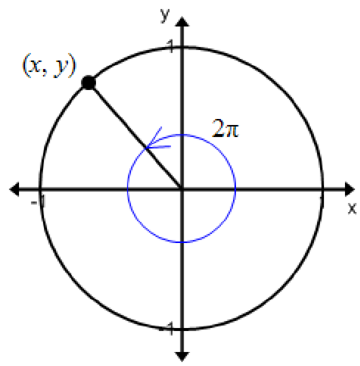
Using symmetry, find exact values of $\sin\theta$ and $\cos\theta$ if $\theta=-\frac{\pi}{3}$.

$\sin(-\theta)=-\sin(\theta)$ $\sin\left(-\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$	Sine is an odd function.
$\cos(-\theta)=\cos(\theta)$ $\cos\left(-\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$	Cosine is an even function.

Periodicity

A function, *f*, is said to be periodic if there is a positive number *P* such that $f(\theta+P)=f(\theta)$ for all θ in the domain. The smallest number *P* for which this occurs is called the period of *f*.

Sine and cosine are periodic functions and have a period of 2π. Any point (x,y) on the unit circle, will be repeated after a rotation of $\pm 2\pi$. Therefore, $(x\pm 2\pi, y\pm 2\pi)$ will be mapped on to (x,y). $\sin(\theta \pm 2n\pi)=\sin(\theta)$ and $\cos(\theta\pm 2n\pi)=\cos(\theta)$, where *n* is any integer.



Example 2:

Using periodicity evaluate the following:

a. $\sin(9\pi)$

b. $\cos\left(\frac{31\pi}{6}\right)$

a. $\sin(9\pi)$	
$\sin(9\pi)=\sin(4(2\pi)+\pi)$	It is obvious that the angle 9π is more than one rotation of the unit circle. It is in fact 4 full rotations and π more.
$\sin(9\pi)=\sin(\pi)=0$	Because of periodicity this statement is true.

b. $\cos\left(\frac{31\pi}{6}\right)$	
$\cos\left(\frac{31\pi}{6}\right)=\cos\left(2(2\pi)+\frac{7\pi}{6}\right)$	One complete rotation of the unit circle in terms of sixths would be $2\pi=\frac{12\pi}{6}$. $\frac{31\pi}{6}$ is at least twice $\left(\frac{24\pi}{6}\right)$ around the unit circle, but not three $\left(\frac{36\pi}{6}\right)$ times around it.
$\cos\left(\frac{31\pi}{6}\right)=\cos\left(\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2}$	Because of periodicity this statement is true.

Practice Exercises A

Use symmetry to find exact values of $\sin\theta$ and $\cos\theta$ for the given angle.

1. $\theta = -\frac{2\pi}{3}$
2. $\theta = -\frac{\pi}{6}$
3. $\theta = -\frac{3\pi}{4}$
4. $\theta = -\frac{5\pi}{3}$
5. $\theta = -\frac{5\pi}{6}$
6. $\theta = -\frac{\pi}{2}$

Use periodicity to evaluate the expression.

7. $\sin\left(\frac{13\pi}{6}\right)$
8. $\cos\left(\frac{14\pi}{3}\right)$
9. $\sin\left(\frac{13\pi}{2}\right)$
10. $\cos\left(\frac{9\pi}{4}\right)$
11. $\sin\left(\frac{61\pi}{3}\right)$
12. $\cos\left(\frac{23\pi}{6}\right)$
13. $\sin\left(\frac{27\pi}{4}\right)$
14. $\cos(20\pi)$
15. $\sin\left(\frac{7\pi}{2}\right)$

Unit 3 Clusters 2 and 3 HONORS (F.TF.6 and F.TF.7): Inverse Trigonometric Functions

Cluster 2: Extending the domain of trigonometric functions using the unit circle

3.3H Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

3.3H Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology and interpret them in terms of the context.

Recall that in order for the inverse of a function to be a function, the original function must be a one-to-one function and meet the criteria for the vertical and horizontal line tests. Not all functions meet the criteria to have an inverse which is also a function. However, if the domain is restricted so that the original function is one-to-one, then the inverse will be a function. Determining which portion of the domain to use can be challenging. Generally, it is best to find an interval of the domain where the function is always increasing or always decreasing.

Graphing Inverse Trigonometric Functions

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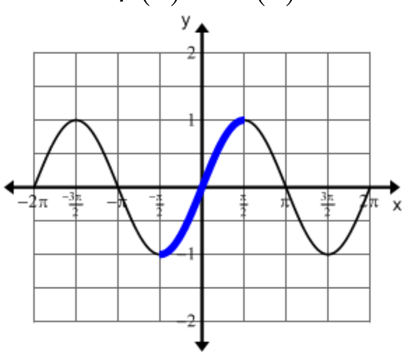
$f(x)=\sin(x)$



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The sine function does not pass the horizontal line test. In order to graph its inverse, the domain will have to be restricted.

The function is increasing on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. By restricting the domain to this interval, it will allow its inverse to be constructed.

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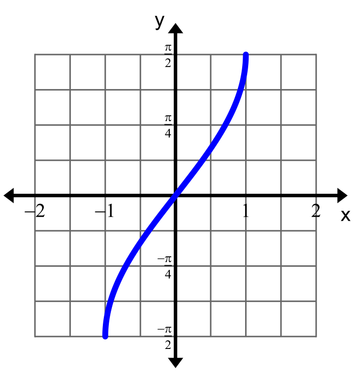
$f(x)=\sin^{-1}(x)$



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The inverse of the sine function (**arcsine function**) is denoted two ways: $f(x)=\sin^{-1}(x)$ or $f(x)=\arcsin(x)$. The *x*-coordinate now represents the sine value and the *y*-coordinate represents the unique angle. The domain is $[-1,1]$ and the range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

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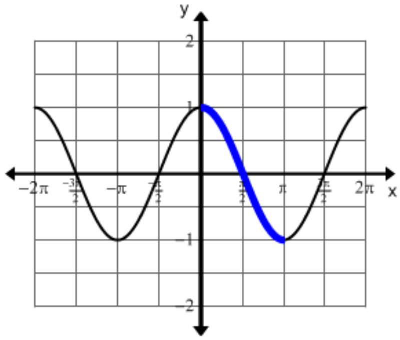
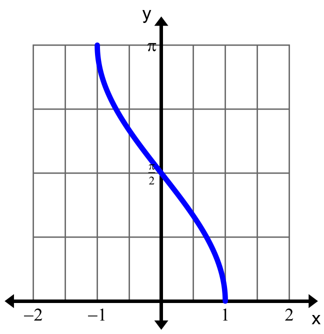
$f(x) = \cos(x)$



The cosine function does not pass the horizontal line test. In order to graph its inverse, the domain will have to be restricted.

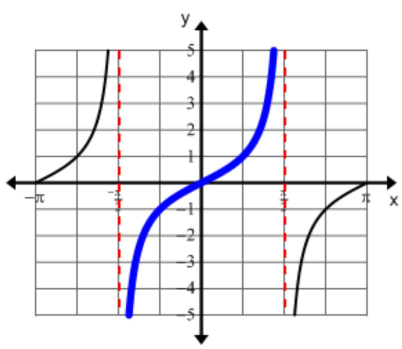
The function is decreasing on the interval $(0, \pi)$. By restricting the domain to this interval, it will allow its inverse to be constructed.

$f(x) = \cos^{-1}(x)$



The inverse of the cosine function (**arccosine function**) is denoted two ways: $f(x) = \cos^{-1}(x)$ or $f(x) = \arccos(x)$. The *x*-coordinate now represents the cosine value and the *y*-coordinate represents the unique angle. The domain is $[-1, 1]$ and the range is $[0, \pi]$.

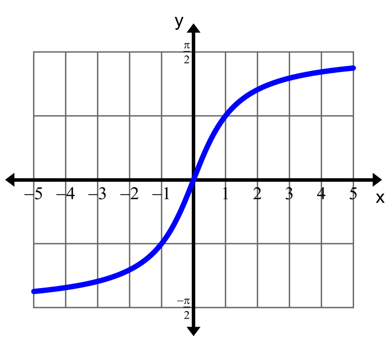
$f(x) = \tan(x)$



The tangent function does not pass the horizontal line test. In order to graph its inverse, the domain will have to be restricted.

The function is increasing on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. By restricting the domain to this interval, it will allow its inverse to be constructed.

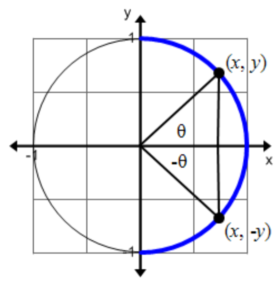
$f(x) = \tan^{-1}(x)$

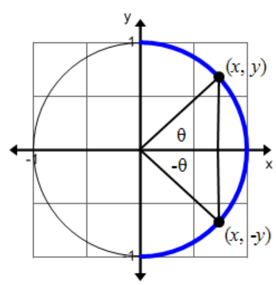


The inverse of the tangent function (**arctangent function**) is denoted two ways: $f(x) = \tan^{-1}(x)$ or $f(x) = \arctan(x)$. The *x*-coordinate now represents the tangent value and the *y*-coordinate represents the unique angle. The domain is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example 1:
Find the exact value of the expression without a calculator.

a. $\sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right)$ b. $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

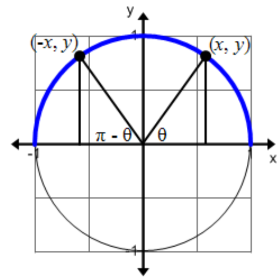
a. $\sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right)$ $\sin\theta=\dfrac{\sqrt{2}}{2}$	This is really asking us to find the angle, θ, that is in the interval $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ whose sine value is $\dfrac{\sqrt{2}}{2}$.
$\sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right)=\dfrac{\pi}{4}$	The inverse sine function is restricted to the first and fourth quadrants of the unit circle. Sine is positive in the first quadrant, therefore, θ will be an angle from the first quadrant. 

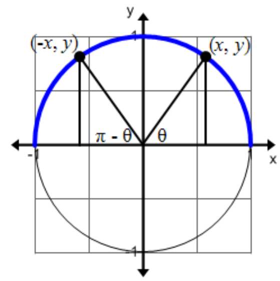
b. $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ $\sin\theta=-\dfrac{\sqrt{3}}{2}$	This is really asking us to find the angle, θ, that is in the interval $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ whose sine value is $-\dfrac{\sqrt{3}}{2}$.
$\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)=-\dfrac{\pi}{3}$	The inverse sine function is restricted to the first and fourth quadrants of the unit circle. Sine is negative in the fourth quadrant, therefore, θ will be an angle from the fourth quadrant. 

Example 2:
Find the exact value of the expression without a calculator.

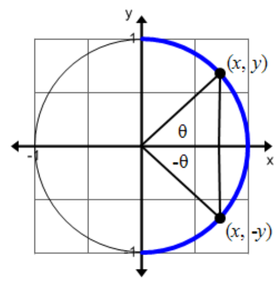
a. $\cos^{-1}\left(\dfrac{1}{2}\right)$ b. $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

a. $\cos^{-1}\left(\dfrac{1}{2}\right)$ $\cos\theta=\dfrac{1}{2}$	This is really asking us to find the angle, θ, that is in the interval $[0,\pi]$ whose cosine value is $\dfrac{1}{2}$.

$\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$	The inverse cosine function is restricted to the first and second quadrants of the unit circle. Cosine is positive in the first quadrant, therefore, θ will be an angle from the first quadrant. 

b. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $\cos\theta=-\frac{\sqrt{2}}{2}$	This is really asking us to find the angle, θ, that is in the interval $[0,\pi]$ whose cosine value is $-\frac{\sqrt{2}}{2}$.
$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4}$	The inverse cosine function is restricted to the first and second quadrants of the unit circle. Cosine is negative in the second quadrant, therefore, θ will be an angle from the second quadrant. 

Example 3:
Find the exact value of the expression $\tan^{-1}(1)$ without using a calculator.

a. $\tan^{-1}(1)$ $\tan\theta=1$	This is really asking us to find the angle, θ, that is in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent value is 1.
$\tan^{-1}(1)=\frac{\pi}{4}$	The inverse tangent function is restricted to the first and fourth quadrants of the unit circle. Tangent is positive in the first quadrant, therefore, θ will be an angle from the first quadrant. 

Practice Exercises A

Find the exact value of the expression without a calculator.

1. $\sin^{-1}\left(\frac{1}{2}\right)$
2. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
3. $\sin^{-1}(1)$
4. $\sin^{-1}\left(-\frac{1}{2}\right)$
5. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
6. $\sin^{-1}(0)$
7. $\sin^{-1}(-1)$
8. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
9. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
10. $\cos^{-1}(1)$
11. $\cos^{-1}(-1)$
12. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
13. $\cos^{-1}\left(-\frac{1}{2}\right)$
14. $\cos^{-1}(0)$
15. $\tan^{-1}\left(\sqrt{3}\right)$
16. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$
17. $\tan^{-1}(0)$
18. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
19. $\tan^{-1}(-1)$
20. $\tan^{-1}\left(-\sqrt{3}\right)$

Compositions Involving Inverse Trigonometric Functions

You may recall that for inverse functions the following properties are true: $f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$. Thus, the following properties are true for trigonometric functions and their inverses:

- $\sin\left(\sin^{-1}(x)\right)=x$ for every x in the interval $[-1,1]$.
- $\sin^{-1}\left(\sin(x)\right)=x$ for every x in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
- $\cos\left(\cos^{-1}(x)\right)=x$ for every x in the interval $[-1,1]$.
- $\cos^{-1}\left(\cos(x)\right)=x$ for every x in the interval $[0,\pi]$.
- $\tan\left(\tan^{-1}(x)\right)=x$ for every x in the interval $(-\infty,\infty)$.
- $\tan^{-1}\left(\tan(x)\right)=x$ for every x in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Example 4:
Find the exact value of the expression.

a. $\sin\left(\sin^{-1}(0.3)\right)$

b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

c. $\tan^{-1}\left(\tan(\pi)\right)$

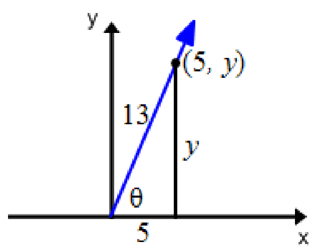
a. $\sin\left(\sin^{-1}(0.3)\right)$ $\sin\left(\sin^{-1}(0.3)\right)=0.3$	$x=0.3$ which is in the interval $[-1,1]$. Use the properties of inverse trigonometric functions.
b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$ $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$	$x=\frac{7\pi}{6}$ which is not in the interval $[0,\pi]$. Evaluate the inside function $\cos\left(\frac{7\pi}{6}\right)$. Find the angle whose cosine value is $-\frac{\sqrt{3}}{2}$.
c. $\tan^{-1}(\tan(\pi))$ $\tan^{-1}(0)$ $\tan^{-1}(0)=0$	$x=\pi$ which is not in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Evaluate the inside function $\tan(\pi)$. Find the angle whose tangent value is 0.

Example 5:
Find the exact value of the expression.

a. $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

b. $\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right)$

a. Using the definition of the inverse cosine function, we can rewrite the inside function as $\cos\theta=\frac{5}{13}$. The cosine value is positive, which means that the angle we are looking for is in the first quadrant, as the picture below demonstrates.

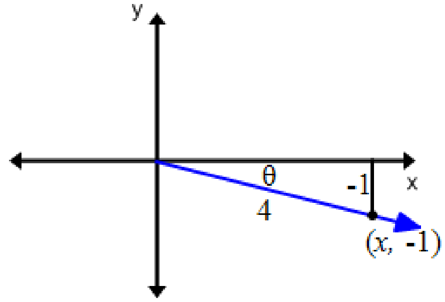


It is not necessary to find the actual value of the angle because $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$ is asking us to find the tangent of the angle whose cosine is $\frac{5}{13}$. Recall that $\tan\theta=\frac{\text{opposite}}{\text{adjacent}}$.

Using the Pythagorean theorem, the opposite side is: $y=\sqrt{13^2-5^2}=\sqrt{169-25}=\sqrt{144}=12$.

$$\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)=\tan(\theta)=\frac{12}{5}$$

b. Using the definition of the inverse sine function, we can rewrite the inside function as $\sin\theta = -\frac{1}{4}$. The sine value is negative, which means that the angle we are looking for is in the fourth quadrant, as the picture below demonstrates.



It is not necessary to find the actual value of the angle because $\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right)$ is asking us to find the cosine of the angle whose sine is $-\frac{1}{4}$. Recall that $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.

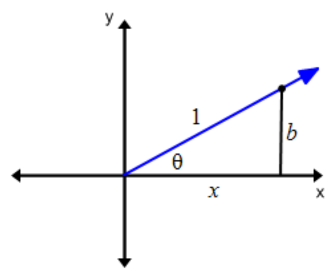
Using the Pythagorean theorem, the adjacent side is: $x = \sqrt{4^2 - (-1)^2} = \sqrt{16-1} = \sqrt{15}$.

$$\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right) = \cos(\theta) = \frac{\sqrt{15}}{4}.$$

Example 6:

Write $\sin\left(\cos^{-1} x\right)$ as an algebraic expression if $0 < x \le 1$.

The expression is limited to the first quadrant by the statement $0 < x \le 1$. Using the definition of the inverse cosine function, we can rewrite the inside function as $\cos\theta = x = \frac{x}{1}$.



It is not necessary to find the actual value of the angle because $\sin\left(\cos^{-1} x\right)$ is asking us to find the sine of the angle whose cosine is $\frac{x}{1}$. Recall that $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$.

Using the Pythagorean theorem, the opposite side is: $b = \sqrt{1^2 - x^2}$.

$$\sin\left(\cos^{-1} x\right) = \sin(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

Practice Exercises B

Find the exact value of the expression.

1. $\sin\left(\sin^{-1}(0.8)\right)$
2. $\cos\left(\cos^{-1}(0.45)\right)$
3. $\tan\left(\tan^{-1}(7)\right)$
4. $\tan\left(\tan^{-1}(100)\right)$
5. $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$
6. $\cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$
7. $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$
8. $\tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)$
9. $\sin^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$
10. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$
11. $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$
12. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
13. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$
14. $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$
15. $\cos^{-1}\left(\cos\left(\frac{11\pi}{6}\right)\right)$

Find the exact value of the expression.

16. $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$
17. $\sin\left(\tan^{-1}(1)\right)$
18. $\tan\left(\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$
19. $\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$
20. $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$
21. $\cos\left(\tan^{-1}\left(\frac{7}{24}\right)\right)$
22. $\tan\left(\sin^{-1}\left(\frac{5}{13}\right)\right)$
23. $\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$
24. $\sin\left(\cos^{-1}\left(-\frac{4}{5}\right)\right)$
25. $\sin\left(\tan^{-1}\left(\frac{15}{8}\right)\right)$
26. $\cos\left(\sin^{-1}\left(\frac{4}{7}\right)\right)$
27. $\tan\left(\sin^{-1}\left(\frac{3}{4}\right)\right)$

Write each expression as an algebraic expression if $0 < x \leq 1$.

28. $\tan\left(\sin^{-1}(x)\right)$
29. $\sin\left(\tan^{-1}(x)\right)$
30. $\sin\left(\cos^{-1}\left(\frac{1}{x}\right)\right)$
31. $\tan\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$
32. $\cos\left(\tan^{-1}\frac{x}{\sqrt{3}}\right)$
33. $\sin\left(\tan^{-1}\left(\frac{x}{2}\right)\right)$
34. $\cos\left(\sin^{-1}(2x)\right)$
35. $\cos\left(\sin^{-1}\left(\frac{\sqrt{x^2-9}}{x}\right)\right)$
36. $\tan\left(\cos^{-1}\left(\frac{x}{\sqrt{x^2+4}}\right)\right)$

Unit 3 Clusters 2 and 3 HONORS: Solving Trigonometric Equations

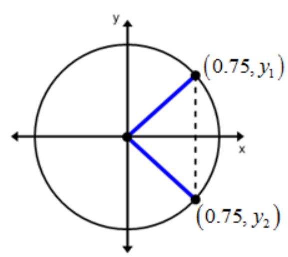
Cluster 2: Extending the domain of trigonometric functions using the unit circle
3.3H Find all solutions for equations involving trigonometric functions.

Example 1:
Find all values of x, $0 \le x < 2\pi$, that make the statement true.

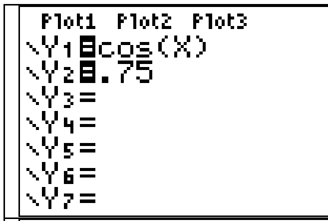
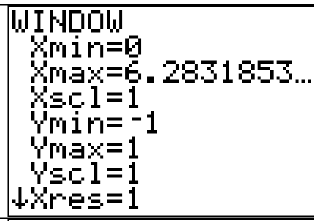
a. $\sin x = \frac{1}{2}$ b. $\cos x = -1$ c. $\csc x = \frac{2}{\sqrt{3}}$

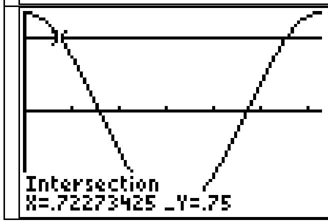
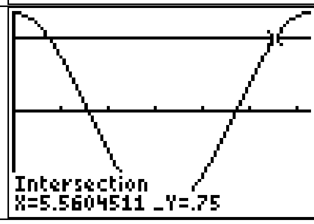
a.	The points that have been rotated $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ radians from the positive *x*-axis have coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ respectively. The *y*-coordinate is the sine value and both points have a *y*-coordinate of $\frac{1}{2}$.
b.	The point that has been rotated π radians from the positive *x*-axis has coordinates $(-1,0)$. The *x*-coordinate is the cosine value which is -1.
c.	Cosecant is the reciprocal of sine, therefore find all the values that satisfy $\sin x = \frac{\sqrt{3}}{2}$. The angles rotated $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ radians from the positive *x*-axis have coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ respectively. Both have *y*-coordinates of $\frac{\sqrt{3}}{2}$. Both will have a cosecant of $\frac{2}{\sqrt{3}}$.

Example 2:
Use technology to find all solutions in the interval $[0, 2\pi)$ to the trigonometric equation $\cos x = 0.75$.

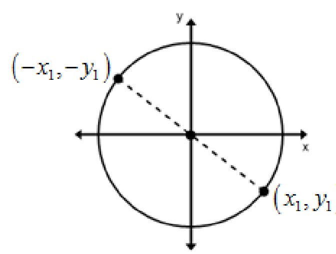
$\cos x = 0.75$	
$x = \cos^{-1}(0.75)$ $x \approx 0.7227$	Make sure that your calculator is in radian mode. Then use the inverse cosine function to find the first value.
$x \approx 2\pi - 0.7227 \approx 5.5605$	The calculator only returns values between 0 and π for inverse cosine. Cosine is positive in the first and fourth quadrants. To get the second value, subtract the first value from 2π. 

This could also be solved graphically.

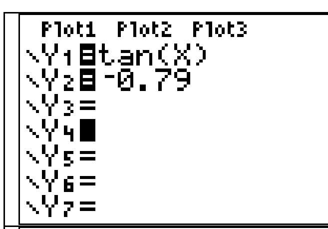
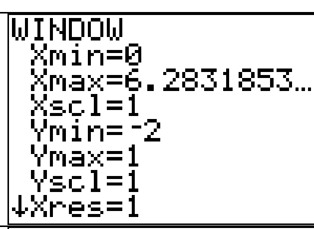
<table>
<tr><td>
</td><td>
</td><td>Enter $y=\cos x$ and $y=0.75$ in your graphing calculator.

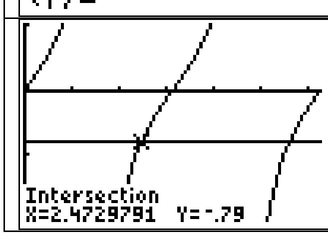
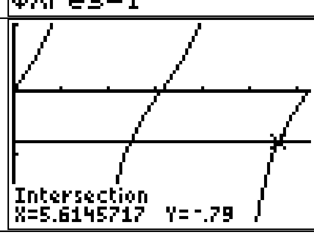
Set your window so that xmin = 0, xmax = 2π , ymin= -1, and ymax = 1.</td></tr>
<tr><td>
</td><td>
</td><td>Graph the functions and find both of the intersections.</td></tr>
</table>

Example 3:
Use technology to find all solutions in the interval $[0,2\pi)$ to the trigonometric equation $\tan x=-0.79$.

<table>
<tr><td>$\tan x=-0.79$</td><td></td></tr>
<tr><td>$x=\tan^{-1}(-0.79)$
$x\approx-0.6686$</td><td>Make sure that your calculator is in radian mode. Then use the inverse tangent function to find the first value.</td></tr>
<tr><td>$x\approx-0.6686+\pi\approx2.4730$</td><td>This is not a value between $[0,2\pi)$. The calculator only returns values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for inverse tangent. You need to add π to your answer. To get the value in the second quadrant.

</td></tr>
<tr><td>$x\approx-0.6686+2\pi\approx5.6146$</td><td>Tangent is negative in the second and fourth quadrants. To get the value in the fourth quadrant, add 2π to the value the calculator gave you.</td></tr>
</table>

This could also be solved graphically.

<table>
<tr><td>
</td><td>
</td><td>Enter $y=\tan x$ and $y=-0.79$ in your graphing calculator.

Set your window so that xmin = 0, xmax = 2π , ymin= -2, and ymax = 1.</td></tr>
<tr><td>
</td><td>
</td><td>Graph the functions and find both of the intersections.</td></tr>
</table>

Practice Exercises A

Find all values of x, $0 \le x < 2\pi$, that make the statement true.

1. $\sin x = -\frac{\sqrt{2}}{2}$
2. $\cos x = \frac{1}{2}$
3. $\tan x = -1$
4. $\sin x = -1$
5. $\cos x = -\frac{\sqrt{3}}{2}$
6. $\tan x = 0$
7. $\sin x = -\frac{\sqrt{3}}{2}$
8. $\cos x = \frac{\sqrt{2}}{2}$
9. $\tan x = \frac{\sqrt{3}}{3}$
10. $\sin x = 0$
11. $\cos x = -1$
12. $\tan x = 1$
13. $\sec x = -\frac{2}{\sqrt{3}}$
14. $\csc x = 2$
15. $\cot x = -1$
16. $\sec x = 1$
17. $\csc x = -1$
18. $\cot x = -\sqrt{3}$

Use technology to find all solutions in the interval $[0, 2\pi)$.

19. $\sin x = 0.33$
20. $\cos x = 0.59$
21. $\tan x = 1.615$
22. $\cos x = -0.36$
23. $\tan x = 0.43$
24. $\sin x = 0.23$
25. $\tan x = -2.4$
26. $\sin x = -0.88$
27. $\cos x = 0.17$

Example 4:
Find all solutions in the interval $[0, 2\pi)$ to the trigonometric equation $2\sin x - 1 = 0$.

$2\sin x - 1 = 0$ $2\sin x = 1$ $\sin x = \frac{1}{2}$	Isolate the trigonometric function.
$x = \sin^{-1}\left(\frac{1}{2}\right)$	Find all angles between 0 and 2π that have a sine value of $\frac{1}{2}$.
$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$	

Example 5:
Find all solutions in the interval $\left[0, 2\pi\right)$ to the trigonometric equation $\sin x \cos x - 3\cos x = 0$.

$\sin x \cos x - 3\cos x = 0$ $\cos x(\sin x - 3) = 0$		Factor out the common term $\cos x$.
$\cos x = 0$	$\sin x - 3 = 0$ $\sin x = 3$	Set each factor equal to zero.
$x = \cos^{-1}\left(0\right)$ $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$	$\sin x \neq 3$	Find all angles between 0 and 2π that have a cosine value of 0. Note: $\sin x$ oscillates between -1 and 1 and will never equal 3.

Example 6:
Find all solutions in the interval $\left[0, 2\pi\right]$ to the trigonometric equation $2\cos^2 x - \cos x - 1 = 0$.

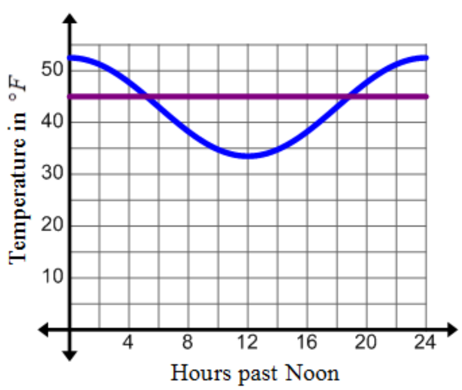
$2\cos^2 x - \cos x - 1 = 0$ $2u^2 - u - 1 = 0$		Let $u = \cos x$ and rewrite the equation.
$\left(2u+1\right)\left(u-1\right) = 0$		Factor the quadratic.
$2u + 1 = 0$ $2u = -1$ $u = -\frac{1}{2}$	$u - 1 = 0$ $u = 1$	Set each factor equal to zero.
$\cos x = -\frac{1}{2}$	$\cos x = 1$	Substitute $\cos x$ back in for *u*.
$x = \cos^{-1}\left(-\frac{1}{2}\right)$ $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$	$x = \cos^{-1}\left(1\right)$ $x = 0$ or $x = 2\pi$	Find all angles between 0 and 2π that have a cosine value of $-\frac{1}{2}$. Find all angles between 0 and 2π inclusive that have a cosine value of 1.

Example 7:
Find all solutions in the interval $\left[0, 2\pi\right)$ to the trigonometric equation $4\cos^2 x \sin x = 3\sin x$.

$4\cos^2 x \sin x = 3\sin x$ $4\cos^2 x \sin x - 3\sin x = 0$ $\sin x\left(4\cos^2 x - 3\right) = 0$	Get all the terms on the same side, then factor out the common term $\sin x$.

$\sin x = 0$	$4\cos^2 x - 3 = 0$ $4\cos^2 x = 3$ $\cos^2 x = \frac{3}{4}$ $\cos x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$	Set each factor equal to zero and isolate the trigonometric expression.
$x = \sin^{-1}(0)$ $x = 0$ or $x = \pi$	$x = \cos^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)$ $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6},$ or $\frac{11\pi}{6}$	Find all angles between 0 and 2π that have a sine value of 0. Find all angles between 0 and 2π that have a cosine value of $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$.

Example 8:
A heater turns on in a home when the outside temperature is below $45°F$. During the middle of March in Salt Lake City, you can model the outside temperature in degrees Fahrenheit using the function $f(t) = 43 + 9.5\cos\frac{\pi}{12}t$, where t is the number of hours past noon. During which hours is the heater heating the home?

	Find where $f(t) = 43 + 9.5\cos\frac{\pi}{12}t$ is equal to $45°F$. The outside temperature is equal to $45°F$ when $t \approx 5.2$ hours past noon and $t \approx 18.8$ hours past noon. 5.2 hours is 5 hours and 12 minutes. $0.2(60) = 12$ 18.8 hours is 18 hours and 48 minutes. $0.8(60) = 48$ The heater is on from 5:12 p.m. to 6:48 a.m.

Practice Exercises B

Find all solutions in the interval $[0, 2\pi)$ to the trigonometric equation.

1. $2\sin x + \sqrt{3} = 0$

2. $3\tan x - \sqrt{3} = 0$

3. $\cos x + \sqrt{2} = -\cos x$

4. $5 + 2\sin x - 7 = 0$

5. $3\cos x = \cos x - 1$

6. $4\sin x = 2\sin x + 1$

7. $4\tan x = 3 + \tan x$

8. $\sqrt{3}\cos x \tan x - \cos x = 0$

9. $2\sin^3 x = \sin x$

10. $\sin x = -\sin x \cos x$

11. $\tan x = \tan^2 x$

12. $2\cos x \sin x - \cos x = 0$

13. $2\sin^2 x - 1 = 0$

14. $4\cos^2 x - 1 = 0$

15. $5\tan^2 x - 15 = 0$

16. $4\cos^2 x - 3 = 0$

17. $\sqrt{2}\tan x \cos x - \tan x = 0$

18. $\sin^2 x - 2\sin x = 0$

19. $\tan x \sin^2 x = \tan x$

20. $3\tan^2 x - 1 = 0$

21. $2\sin^2 x + 3\sin x + 1 = 0$

22. $4\cos^2 x - 4\cos x + 1 = 0$

23. $\sin^2 x + 3\sin x = 0$

24. $2\sin^2 x - 3\sin x = 2$

25. The function $I = 40\sin(60\pi t)$ models the current, *I*, in amps that an electric generator is producing after *t* seconds. How many seconds is the current above 25 amps?

26. The function $h = 25\sin\left(\frac{\pi}{20}(t-10)\right) + 34$ models the height, *h*, of a Ferris wheel car in feet, *t* seconds after starting. On the first rotation only, when is the Ferris wheel 50 feet above ground?

27. The water level of a harbor can be modeled by the equation $f(t) = -30\cos\left(\frac{6\pi}{37}t\right)$, where *t* represents the hours after low tide and *f* is the water depth in feet. Determine how many hours after low tide the water level is at 15 feet during the day.

28. The intensity of a sound wave for a certain pitch fork can be modeled by the function $f(t) = 0.001\sin(1320\pi t)$, where *t* is measured in seconds. When does the intensity first reach -0.0006?

Polar and Parametric

Honors Unit: Polar Coordinates and Equations

Honors Cluster: Polar and Parametric

H1: Define and use polar coordinates and relate them to Cartesian coordinates.

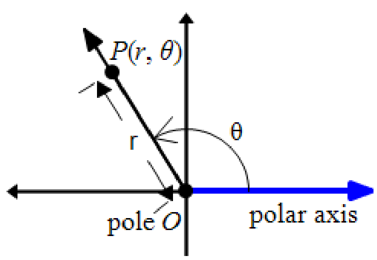
H3: Translate equations in Cartesian coordinates into polar coordinates and graph equations in the polar coordinate plane.

Polar Coordinates

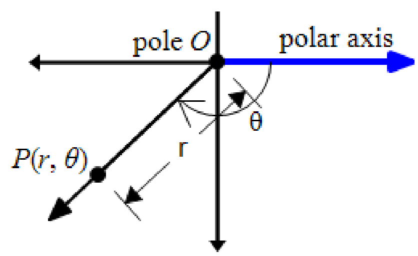
VOCABULARY

In addition to the Cartesian plane and the complex plane, there is another coordinate plane called the **polar coordinate system**. The origin of the polar coordinate system, *O*, is called the **pole**. The pole is the endpoint of a ray, called the **polar axis**, which extends to the right. A point, *P*, on the polar coordinate system is represented by (r,θ), where *r* is the directed distance from the pole, *O*, to point *P* and θ is the angle of rotation with initial side the polar axis and terminal side the segment OP. Just like in trigonometry, positive angles rotate counterclockwise from the polar axis and negative angles rotate clockwise from the polar axis.

Positive Rotation



Negative Rotation

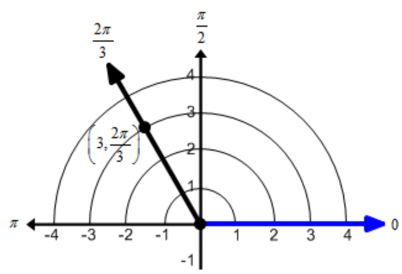


Example 1:

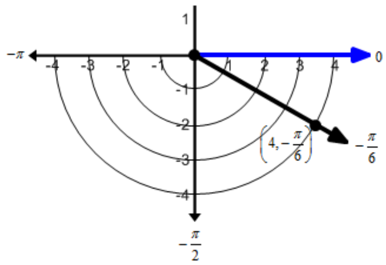
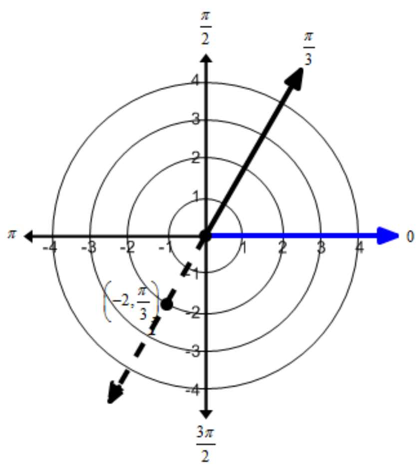
Plot the following points on the polar coordinate system.

a. $\left(3,\frac{2\pi}{3}\right)$ b. $\left(4,-\frac{\pi}{6}\right)$ c. $\left(-2,\frac{\pi}{3}\right)$

a. $\left(3,\frac{2\pi}{3}\right)$



Rotate the ray on the polar axis $\frac{2\pi}{3}$ radians counterclockwise and go out 3 units from the pole.

b. $\left(4, -\frac{\pi}{6}\right)$ 	Rotate the ray on the polar axis $\frac{\pi}{6}$ radians clockwise and go out 4 units from the pole.
c. $\left(-2, \frac{\pi}{3}\right)$ 	Rotate the ray on the polar axis $\frac{\pi}{3}$ radians counterclockwise and go out 2 units on the ray opposite the terminal side of the angle.

Unlike rectangular coordinates, (x, y), that have exactly one representation, polar coordinates can be represented in infinitely many ways. Take a closer look at Example 1 part c. The point $\left(-2, \frac{\pi}{3}\right)$ could also have been written as $\left(2, \frac{4\pi}{3}\right)$ or even $\left(-2, \frac{7\pi}{3}\right)$. To get the point $\left(2, \frac{4\pi}{3}\right)$ from the point $\left(-2, \frac{\pi}{3}\right)$, r was changed to a positive number and π was added to θ. To get the point $\left(-2, \frac{7\pi}{3}\right)$ from the point $\left(-2, \frac{\pi}{3}\right)$, 2π was added to θ. The table below summarizes how multiple representations of the same polar point can be found.

Finding Multiple Representations of Points
The point (r, θ) can be represented as $(r, \theta + 2n\pi)$ or $(-r, \theta + \pi + 2n\pi) = (-r, \theta + (2n+1)\pi)$, where n is an integer.

Example 2:
For each polar coordinate, find another representation in which:

1. r is positive and θ is on the interval $(2\pi, 4\pi)$.
2. r is negative and θ is on the interval $(0, 2\pi)$.
3. r is positive and θ is on the interval $(-2\pi, 0)$.

a. $\left(3, \frac{\pi}{4}\right)$

b. $\left(5, \frac{2\pi}{3}\right)$

a.

1. $\left(3, \frac{\pi}{4}\right) = \left(3, \frac{\pi}{4} + 2\pi\right) = \left(3, \frac{9\pi}{4}\right)$ Add 4π to θ.
2. $\left(3, \frac{\pi}{4}\right) = \left(-3, \frac{\pi}{4} + \pi\right) = \left(-3, \frac{5\pi}{4}\right)$ Change the sign of r and add π to θ.
3. $\left(3, \frac{\pi}{4}\right) = \left(3, \frac{\pi}{4} - 2\pi\right) = \left(3, -\frac{7\pi}{4}\right)$ Subtract 2π from θ.

b.

1. $\left(5, \frac{2\pi}{3}\right) = \left(5, \frac{2\pi}{3} + 2\pi\right) = \left(5, \frac{8\pi}{3}\right)$ Add 2π to θ.
2. $\left(5, \frac{2\pi}{3}\right) = \left(-5, \frac{2\pi}{3} + \pi\right) = \left(-5, \frac{5\pi}{3}\right)$ Change the sign of r and add π to θ.
3. $\left(5, \frac{2\pi}{3}\right) = \left(5, \frac{2\pi}{3} - 2\pi\right) = \left(5, -\frac{4\pi}{3}\right)$ Subtract 2π from θ.

Relating Polar and Cartesian Coordinates

A polar point $P(r, \theta)$ is equal to a rectangular point $P(x, y)$ when the pole is the origin and the polar axis is the positive x-axis. Using trigonometry, equations can be derived that relate the polar coordinates to the rectangular coordinates of a point P.

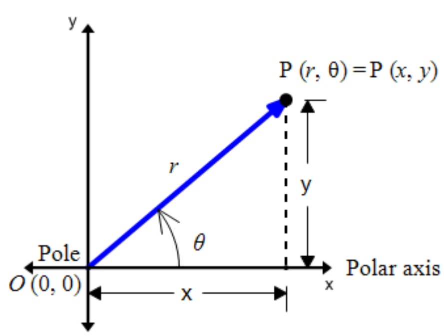
Relating Polar and Cartesian (Rectangular) Coordinates

$\cos\theta = \frac{x}{r}$ therefore $x = r\cos\theta$

$\sin\theta = \frac{y}{r}$ therefore $y = r\sin\theta$

$\tan\theta = \frac{y}{x}$ therefore $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r^2 = x^2 + y^2$ by the Pythagorean Theorem.



Example 3:
Find the rectangular coordinates of the given polar coordinates.

a. $\left(2, \frac{5\pi}{6}\right)$ b. $\left(-4, \frac{4\pi}{3}\right)$

a. $\left(2, \frac{5\pi}{6}\right)$	$r = 2$ and $\theta = \frac{5\pi}{6}$
$x = r\cos\theta$ $x = 2\cos\frac{5\pi}{6}$ $x = 2\left(-\frac{\sqrt{3}}{2}\right)$ $x = -\sqrt{3}$ $y = r\sin\theta$ $y = 2\sin\frac{5\pi}{6}$ $y = 2\left(\frac{1}{2}\right)$ $y = 1$	Use $x = r\cos\theta$ and $y = r\sin\theta$ to find the rectangular coordinates.
$\left(2, \frac{5\pi}{6}\right)$ in rectangular coordinates is $\left(-\sqrt{3}, 1\right)$.	

b. $\left(-4, \frac{4\pi}{3}\right)$	$r = -4$ and $\theta = \frac{4\pi}{3}$
$x = r\cos\theta$ $x = -4\cos\frac{4\pi}{3}$ $x = -4\left(-\frac{\sqrt{2}}{2}\right)$ $x = 2\sqrt{2}$ $y = r\sin\theta$ $y = -4\sin\frac{4\pi}{3}$ $y = -4\left(-\frac{\sqrt{2}}{2}\right)$ $y = 2\sqrt{2}$	Use $x = r\cos\theta$ and $y = r\sin\theta$ to find the rectangular coordinates.
$\left(-4, \frac{4\pi}{3}\right)$ in rectangular coordinates is $\left(2\sqrt{2}, 2\sqrt{2}\right)$.	

Example 4:
Find the polar coordinates of the given rectangular coordinates.

a. $(3, -3)$ b. $\left(-6, -2\sqrt{3}\right)$

a. $(3, -3)$ $r^2 = (3)^2 + (-3)^2$ $r^2 = 9 + 9$ $r^2 = 18$ $r = \pm\sqrt{18}$ $r = 3\sqrt{2}$	Use $r^2 = x^2 + y^2$ to find the radius. Use the positive value for *r*.

$\theta = \tan^{-1}\left(\frac{-3}{3}\right)$ $\theta = \tan^{-1}(-1)$ $\theta = \frac{7\pi}{4}$	Use $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ to find θ. Find the positive angle that is in the fourth quadrant (the same quadrant as the rectangular point).
$(3,-3)$ in polar coordinates is $\left(3\sqrt{2}, \frac{7\pi}{4}\right)$.	

b. $\left(-6,-2\sqrt{3}\right)$ $r^2 = (-6)^2 + \left(-2\sqrt{3}\right)^2$ $r^2 = 36+12$ $r^2 = 48$ $r = \pm\sqrt{48}$ $r = 4\sqrt{3}$	Use $r^2 = x^2 + y^2$ to find the radius. Use the positive value for r.
$\theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-6}\right)$ $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ $\theta = \frac{7\pi}{6}$	Use $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ to find θ. Find the positive angle that is in the third quadrant (the same quadrant as the rectangular point).
$\left(-6,-2\sqrt{3}\right)$ in polar coordinates is $\left(4\sqrt{3}, \frac{7\pi}{6}\right)$.	

Practice Exercises A

Plot the following points on the polar coordinate system.

1. $\left(3, \frac{4\pi}{3}\right)$
2. $\left(2, \frac{5\pi}{6}\right)$
3. $\left(-1, \frac{3\pi}{4}\right)$
4. $(-4, \pi)$
5. $\left(5, -\frac{7\pi}{6}\right)$
6. $\left(-2, -\frac{\pi}{3}\right)$
7. $(-1, 0)$
8. $\left(\frac{3}{2}, \frac{3\pi}{2}\right)$

For each polar coordinate, find another representation in which:

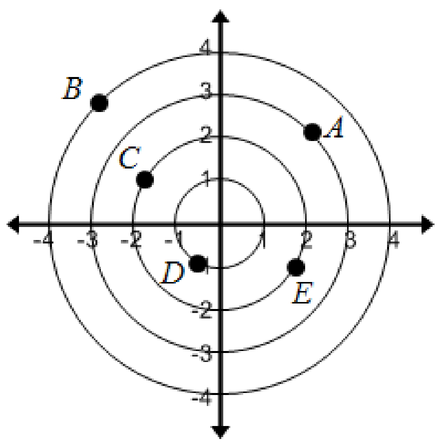
a. r is positive and θ is on the interval $[2\pi, 4\pi)$.

b. r is negative and θ is on the interval $[0, 2\pi)$.

c. r is positive and θ is on the interval $[-2\pi, 0)$.

9. $\left(5, \frac{\pi}{6}\right)$
10. $(7, \pi)$
11. $\left(10, \frac{3\pi}{4}\right)$
12. $\left(2, \frac{4\pi}{3}\right)$
13. $\left(12, \frac{5\pi}{4}\right)$
14. $\left(8, \frac{5\pi}{6}\right)$
15. $\left(3, \frac{\pi}{2}\right)$
16. $\left(4, \frac{5\pi}{3}\right)$

Identify the letter that represents the given polar coordinate.



17. $\left(-1, \frac{\pi}{3}\right)$
18. $\left(-4, -\frac{\pi}{4}\right)$
19. $\left(3, -\frac{7\pi}{4}\right)$
20. $\left(2, -\frac{\pi}{6}\right)$
21. $\left(4, -\frac{5\pi}{4}\right)$
22. $\left(1, \frac{10\pi}{3}\right)$
23. $\left(-2, \frac{11\pi}{6}\right)$
24. $\left(-3, -\frac{3\pi}{4}\right)$

Find the rectangular coordinates of the given polar coordinates.

25. $\left(4, \frac{\pi}{2}\right)$
26. $(6, \pi)$
27. $\left(2, \frac{\pi}{3}\right)$
28. $\left(4, \frac{5\pi}{6}\right)$
29. $\left(-6, \frac{3\pi}{4}\right)$
30. $\left(3, \frac{3\pi}{2}\right)$
31. $\left(5, \frac{7\pi}{6}\right)$
32. $(-3, 0)$

Find the polar coordinates of the given rectangular coordinates.

33. $(-2, 2)$
34. $(2, -2)$
35. $\left(2, -2\sqrt{3}\right)$
36. $\left(-2\sqrt{3}, 2\right)$
37. $\left(-\sqrt{3}, -1\right)$
38. $\left(-1, -\sqrt{3}\right)$
39. $(-3, 0)$
40. $(0, 4)$

Translating Equations in Cartesian Coordinates to Polar Coordinates

The variables of a polar equation are r and θ. When converting a Cartesian equation to a polar equation use the conversion equations: $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$.

Example 5:
Convert the following Cartesian equations to polar equations.

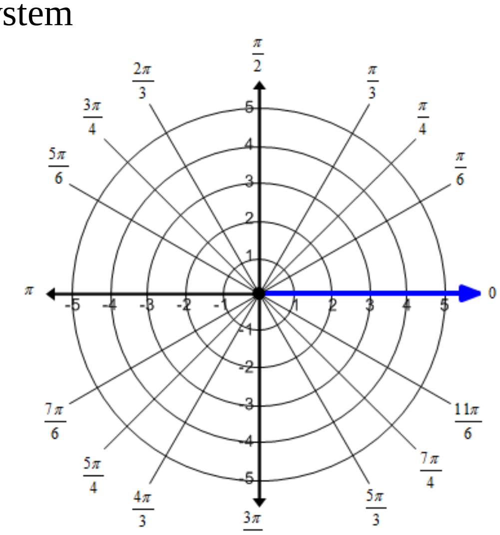
a. $x = 5$ b. $-3x + 4y = 6$ c. $x^2 + (y-4)^2 = 16$

a. $x = 5$ $r\cos\theta = 5$ $r = \dfrac{5}{\cos\theta}$	Substitute $x = r\cos\theta$ then solve the equation for r.
b. $-3x + 4y = 6$ $-3r\cos\theta + 4r\sin\theta = 6$ $r(-3\cos\theta + 4\sin\theta) = 6$ $r = \dfrac{6}{-3\cos\theta + 4\sin\theta}$	Substitute $x = r\cos\theta$ and $y = r\sin\theta$ then solve the equation for r.
c. $x^2 + (y-4)^2 = 16$ $x^2 + y^2 - 8y + 16 = 16$ $x^2 + y^2 - 8y = 0$ $r^2 - 8r\sin\theta = 0$ $r(r - 8\sin\theta) = 0$ $r = 0$ $r - 8\sin\theta = 0$ $r = 8\sin\theta$	Expand the equation and simplify. Substitute $x^2 + y^2 = r^2$ and $y = r\sin\theta$ then solve the equation for r. Set each factor equal to zero. $r = 0$ is a point at the pole. The equation $r = 8\sin\theta$ passes through this point so $r = 8\sin\theta$ is the only equation needed.

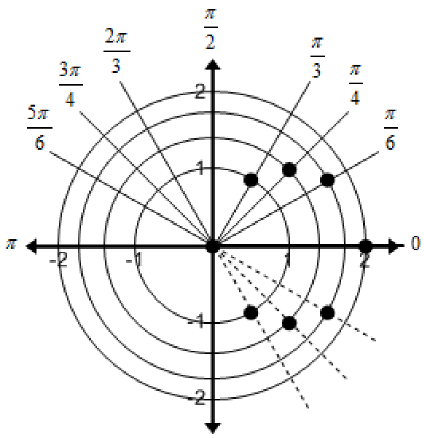
Graphing Equations on the Polar Coordinate System

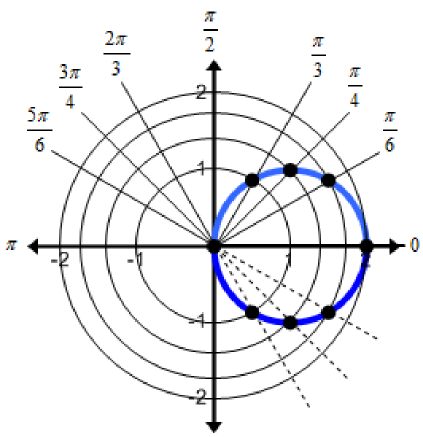
Graphing on the polar coordinate system is like graphing on a set of concentric circles whose center is the pole. The independent variable is θ and the dependent variable is r. The graph is made by plotting all the points that are solutions to the equation.

The graph at the right shows lines going through the pole at angle values for which exact trigonometric values are known. Using these angles makes it easier to graph the polar equations by hand.



Example 6:
Graph the equation $r = 2\cos\theta$.





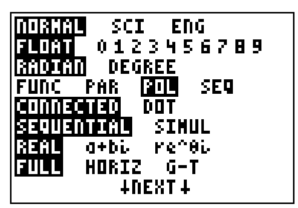
Evaluate the equation at each value of θ then plot the points on the polar coordinate system.

θ	$r = 2\cos\theta$
0	2
$\frac{\pi}{6}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	-1
$\frac{3\pi}{4}$	$-\sqrt{2}$
$\frac{5\pi}{6}$	$-\sqrt{3}$
π	-2

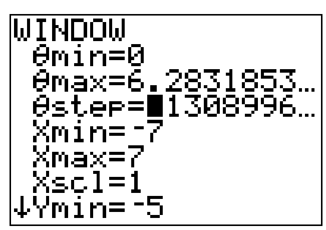
Connect the points in the order that they were plotted.

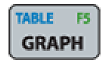
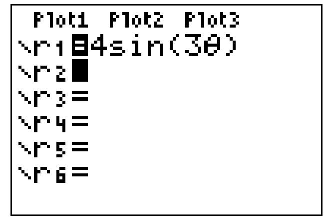
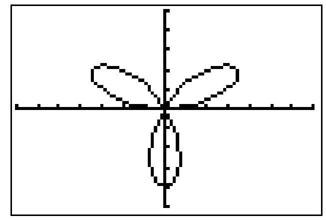
Example 7:
Use technology to graph the equation $r = 4\sin(3\theta)$.

First you will need to change the mode on your calculator so that you are in polar mode (make sure that radians is selected). Push QUIT MODE then arrow down to FUNC PAR POL SEQ. Use your arrow keys to arrow over to POL and push ENTRY SOLVE ENTER so that polar is selected. Exit the screen by pushing 2ND QUIT MODE.



Next you will need to set the window for your equation. Push TBLSET F2 WINDOW. There are a few new items when graphing in polar mode. Typically θ will need to have a minimum of 0 and a maximum of 2π. Set the x minimum to -7 and the x maximum to 7. Set the y minimum to -5 and the y maximum to 5. Exit the screen by pushing 2ND QUIT MODE.



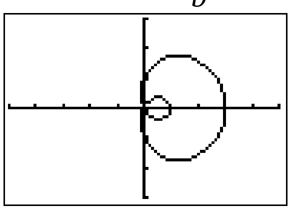
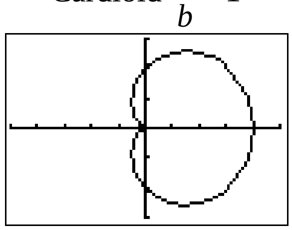
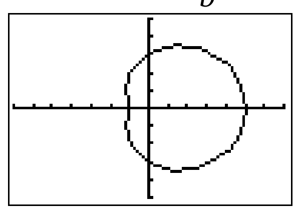
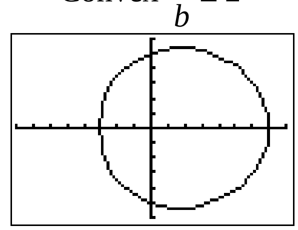
Enter the equation by pushing  . The variable button  will now be θ. Once you have entered the equation, push  and you should get something that looks like the graph at the right.	 

Types of Polar Graphs

Limacons

Equations: $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$ where $a > 0$, $b > 0$

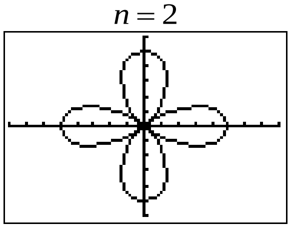
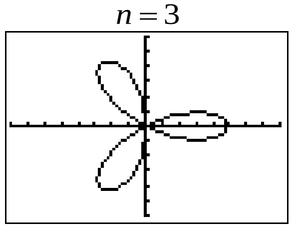
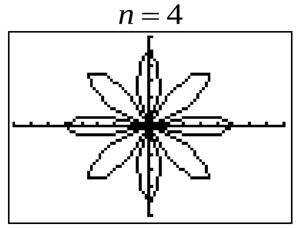
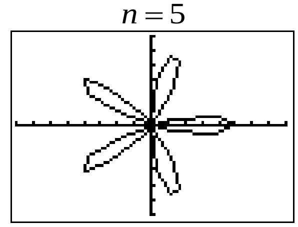
Graphs:

Inner Loop $\frac{a}{b} < 1$	Cardioid $\frac{a}{b} = 1$	Dimpled $1 < \frac{a}{b} < 2$	Convex $\frac{a}{b} \geq 2$
			

Rose Curves

Equations: $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ where $n > 1$ is an integer and $a \neq 0$. The number of petals is determined by the value of *n*. If *n* is even then there will be 2*n* petals. If *n* is odd then there will be *n* petals.

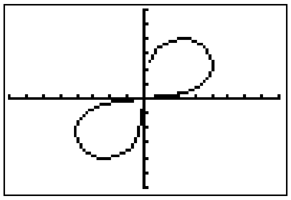
Graphs:

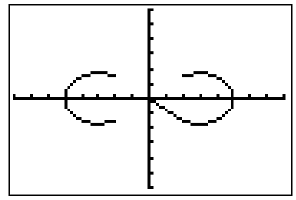
$n = 2$	$n = 3$	$n = 4$	$n = 5$
			

Lemniscates

Equations: $r^2 = a^2 \sin 2\theta$ or $r^2 = a^2 \cos 2\theta$, $a \neq 0$

Graphs:





Practice Exercises B

Convert each Cartesian equation to a polar equation.

1. $x = -3$
2. $x = 5$
3. $y = 4$
4. $y = -8$
5. $2x - 3y = 5$
6. $3x + 4y = 2$
7. $x + 5y = 8$
8. $xy = 4$
9. $x^2 + y^2 = 25$
10. $x^2 + y^2 = 4y$
11. $x^2 + y^2 = 6x$
12. $y^2 = 8x$
13. $x^2 = 3y$
14. $(y+1)^2 = 2x + 1$
15. $(x+4)^2 = 4y + 16$
16. $(x-2)^2 + y^2 = 4$
17. $(x+3)^2 + (y+3)^2 = 18$
18. $(x-1)^2 + (y+4)^2 = 17$

Use technology to graph the following equations on the polar coordinate system.

19. $r = 3\sin\theta$
20. $r = 4\cos\theta$
21. $r = 1 - \sin\theta$
22. $r = 1 + \sin\theta$
23. $r = 2 + \cos\theta$
24. $r = 2 - \sin\theta$
25. $r = 2 - 3\sin\theta$
26. $r = 4\sin 2\theta$
27. $r = 5\cos 3\theta$
28. $r = 2\sin 4\theta$
29. $r^2 = 9\cos 2\theta$
30. $r^2 = 9\sin 2\theta$

Honors Unit: Polar Coordinates and Complex Numbers

H2: Represent complex numbers in rectangular and polar form and convert between rectangular and polar form.
H4: Multiply complex numbers in polar form and use DeMoivre's Theorem to find roots of complex numbers.

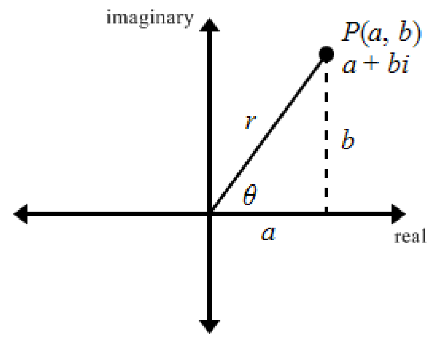
Polar Form of a Complex Number

VOCABULARY

The **absolute value** of a complex number, $a+bi$, is
$|z|=\sqrt{a^2+b^2}$.

The **polar form** (sometimes called trigonometric form) of a complex number is represented by $z=r(\cos\theta+i\sin\theta)$ where $a=r\cos\theta$, $b=r\sin\theta$, r is the modulus (absolute value) of a complex number, and θ is an argument of z.

An **argument** of a complex number is the direction angle of the vector $\langle a,b\rangle$. The angle, θ, can be found using
$\theta=\tan^{-1}\left(\frac{b}{a}\right)$.



Example 1:
Convert the complex numbers in rectangular form to polar form with θ on the interval $[0,2\pi)$.

a. $z=-1+\sqrt{3}i$ b. $z=3-4i$

a. $z=-1+\sqrt{3}i$ $r=\lvert z\rvert=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}$ $r=\sqrt{1+3}$ $r=\sqrt{4}$ $r=2$	Find the modulus or absolute value. $a=-1$ $b=\sqrt{3}$
$\theta=\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ $\theta=\tan^{-1}\left(-\sqrt{3}\right)$ $\theta=\frac{2\pi}{3}$	Find θ using $\theta=\tan^{-1}\left(\frac{b}{a}\right)$. The angle should be in the second quadrant.

$z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$	Use $z = r(\cos\theta + i\sin\theta)$ and substitute in known values.
b. $z = 3 - 4i$ $r = \lvert z\rvert = \sqrt{(3)^2 + (-4)^2}$ $r = \sqrt{9+16}$ $r = \sqrt{25}$ $r = 5$	Find the modulus or absolute value. $a = 3 \;\; b = -4$
$\theta = \tan^{-1}\left(\frac{-4}{3}\right)$ $\theta \approx -0.927$ $\theta \approx -0.927 + 2\pi$ $\theta \approx 5.356$	Find θ using $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. The angle should be in the fourth quadrant. Add 2π to the answer your calculator gives you.
$z = 5(\cos 5.356 + i\sin 5.356)$	Use $z = r(\cos\theta + i\sin\theta)$ and substitute in known values.

Example 2:
Write each complex number in rectangular form.

a. $z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ b. $z = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

a. $z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ $z = 4\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$ $z = -2\sqrt{3} + 2i$	Evaluate cosine and sine at the angle. Distribute the modulus.

b. $z = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ $z = 3(0 + i(1))$ $z = 0 + 3i$	Evaluate cosine and sine at the angle. Distribute the modulus.

Multiplying Complex Numbers in Polar Form

The Product of Two Complex Numbers in Polar Form

To multiply two complex numbers, multiply the moduli and add the arguments. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then the product is $z_1 \cdot z_2 = r_1 \cdot r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$.

Example 3:
Multiply the complex numbers in polar form, then write the result in standard form.

a. $z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z_2 = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

b. $z_1 = 5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$z_2 = 4\left(\cos\pi + i\sin\pi\right)$

a. $z_1 \cdot z_2 = 2\cdot 3\left[\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)\right]$ $z_1 \cdot z_2 = 6\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$	Multiply the moduli and add the arguments.
$z_1 \cdot z_2 = 6\left[-0.966 + i\left(0.259\right)\right]$ $z_1 \cdot z_2 \approx -5.796 + 1.553i$	Evaluate cosine and sine at the angle. Distribute the modulus.

b. $z_1 \cdot z_2 = 5\cdot 4\left[\cos\left(\frac{\pi}{2}+\pi\right) + i\sin\left(\frac{\pi}{2}+\pi\right)\right]$ $z_1 \cdot z_2 = 20\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]$	Multiply the moduli and add the arguments.
$z_1 \cdot z_2 = 20\left[0 + i\left(-1\right)\right]$ $z_1 \cdot z_2 = -20i$	Evaluate cosine and sine at the angle. Distribute the modulus.

Finding Roots of Complex Numbers

De Moivre's Theorem for Complex Roots

Let n be an integer greater than 1 and $z = r\left(\cos\theta + i\sin\theta\right)$ be any complex number. Then z has n distinct nth roots as follows:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$

Example 4:
Find the 5 complex roots of $1+i$. Write the roots in polar form, with θ in radians.

$r=\sqrt{1^2+1^2}=\sqrt{2}$, $\theta=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4}$ $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$	Change $1+i$ into polar form.
1st Root $k=0$ $z_0=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(0)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(0)\pi}{5}\right)\right]$ $z_0=\sqrt[10]{2}\left(\cos\frac{\pi}{20}+i\sin\frac{\pi}{20}\right)$	$\theta=\frac{\pi}{4}$, $n=5$, and $\sqrt[5]{\sqrt{2}}=\sqrt[10]{2}$. Substitute in known information. Simplify.
2nd Root $k=1$ $z_1=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(1)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(1)\pi}{5}\right)\right]$ $z_1=\sqrt[10]{2}\left(\cos\frac{9\pi}{20}+i\sin\frac{9\pi}{20}\right)$	
3rd Root $k=2$ $z_2=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(2)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(2)\pi}{5}\right)\right]$ $z_2=\sqrt[10]{2}\left(\cos\frac{17\pi}{20}+i\sin\frac{17\pi}{20}\right)$	
4th Root $k=3$ $z_3=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(3)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(3)\pi}{5}\right)\right]$ $z_3=\sqrt[10]{2}\left(\cos\frac{25\pi}{20}+i\sin\frac{25\pi}{20}\right)$ $z_3=\sqrt[10]{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$	
5th Root $k=4$ $z_4=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(4)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(4)\pi}{5}\right)\right]$ $z_4=\sqrt[10]{2}\left(\cos\frac{33\pi}{20}+i\sin\frac{33\pi}{20}\right)$	

Example 5:
Find the complex cube roots of 27. Write the roots in rectangular form.

$r=\sqrt{27^2+0^2}=27$, $\theta=\tan^{-1}\left(\dfrac{0}{27}\right)=0$ $27=27(\cos 0+i\sin 0)$	Change 27 into polar form.
1^{st} Root $k=0$ $z_0=3\left[\cos\left(\dfrac{0+2(0)\pi}{3}\right)+i\sin\left(\dfrac{0+2(0)\pi}{3}\right)\right]$ $z_0=3(\cos 0+i\sin 0)$ $z_0=3(1+i\cdot 0)=3$	$\theta=0$, $n=3$, and $\sqrt[3]{27}=3$. Substitute in known information. Simplify.
2^{nd} Root $k=1$ $z_1=3\left[\cos\left(\dfrac{0+2(1)\pi}{3}\right)+i\sin\left(\dfrac{0+2(1)\pi}{3}\right)\right]$ $z_1=3\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$ $z_1=3\left(-\dfrac{1}{2}+i\cdot\dfrac{\sqrt{3}}{2}\right)=-\dfrac{3}{2}+i\dfrac{3\sqrt{3}}{2}$	
3^{rd} Root $k=2$ $z_2=3\left[\cos\left(\dfrac{0+2(2)\pi}{3}\right)+i\sin\left(\dfrac{0+2(2)\pi}{3}\right)\right]$ $z_2=3\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right)$ $z_2=3\left(-\dfrac{1}{2}+i\cdot-\dfrac{\sqrt{3}}{2}\right)=-\dfrac{3}{2}-i\dfrac{3\sqrt{3}}{2}$	

Practice Exercises A

Convert the complex numbers in rectangular form to polar form with θ on the interval $[0,2\pi)$.

1. $3+3i$	2. $1+\sqrt{3}i$	3. $-2-2i$	4. $3-3i$
5. $-5i$	6. $2\sqrt{3}-2i$	7. $-2+2\sqrt{3}i$	8. -4
9. $-3\sqrt{3}+3i$	10. $-4+3i$	11. $-2-3i$	12. $1-\sqrt{5}i$

Write each complex number in rectangular form.

13. $4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

14. $6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

15. $8\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$

16. $10\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

17. $4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

18. $2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

19. $7\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

20. $5\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

21. $3(\cos\pi+i\sin\pi)$

Multiply the complex numbers in polar form, then write the result in standard form. If necessary, round to three decimal places.

22. $z_1=5\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$z_2=3\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

23. $z_1=\sqrt{3}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

$z_2=\frac{1}{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

24. $z_1=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$z_2=4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

25. $z_1=2\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

$z_2=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

26. $z_1=3(\cos\pi+i\sin\pi)$

$z_2=8\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

27. $z_1=\frac{3}{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

$z_2=2\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$

28. $z_1=7\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)$

$z_2=2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

29. $z_1=\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$z_2=\frac{1}{2}\left(\cos\left(-\frac{\pi}{9}\right)+i\sin\left(-\frac{\pi}{9}\right)\right)$

Find the complex roots of each complex number. Write the roots in polar form, with θ in radians.

30. $1-i$, $n=4$

31. $\sqrt{3}-i$, $n=3$

32. $2+2\sqrt{3}i$, $n=5$

33. $-4+4i$, $n=3$

34. $-2\sqrt{3}-2i$, $n=5$

35. $9+3\sqrt{3}i$, $n=4$

Find the complex roots of each number. Write the roots in rectangular form.

36. 64, $n=3$

37. 8, $n=3$

38. 32, $n=5$

Honors Unit: Parametric Equations

H5: Define a curve parametrically and draw parametric graphs.

VOCABULARY

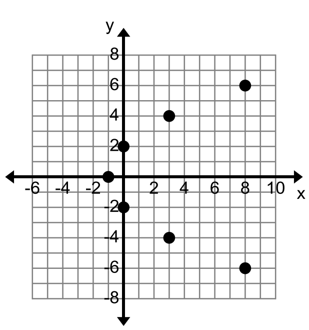
A **parametric curve** is the graph of the ordered pairs (x, y) where $x = f(t)$ and $y = g(t)$. $f(t)$ and $g(t)$ are functions defined on an interval *I*, the **parameter interval**, of *t*-values. The variable *t* is called a **parameter**, and the equations $x = f(t)$ and $y = g(t)$ are **parametric equations** for the curve. The parameter *t* often represents time. The **orientation** of a parametric curve is the direction the curve moves in as *t* increases. Little arrows are drawn on the graph to indicate the orientation.

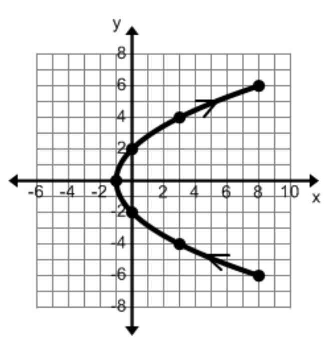
Example 1:
Graph the parametric curve defined by the parametric equations.

$$x = t^2 - 1,\ y = 2t,\ -3 \le t \le 3$$

t	$x = t^2 - 1$	$y = 2t$	(x, y)
-3	$(-3)^2 - 1 = 8$	$2(-3) = -6$	$(8, -6)$
-2	$(-2)^2 - 1 = 3$	$2(-2) = -4$	$(3, -4)$
-1	$(-1)^2 - 1 = 0$	$2(-1) = -2$	$(0, -2)$
0	$(0)^2 - 1 = -1$	$2(0) = 0$	$(-1, 0)$
1	$(1)^2 - 1 = 0$	$2(1) = 2$	$(0, 2)$
2	$(2)^2 - 1 = 3$	$2(2) = 4$	$(3, 4)$
3	$(3)^2 - 1 = 8$	$2(3) = 6$	$(8, 6)$

Construct a table of values.





Plot the points.

Connect the points with a smooth curve indicating the orientation of the curve.

Example 2:
Use technology to graph the curve defined parametrically.

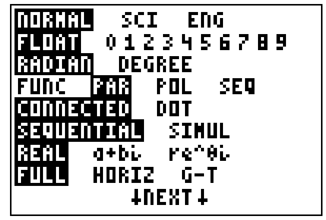
$$x = \sin t - t\cos t,\ \ y = \cos t + t\sin t,\ \ 0 \le t \le 12$$

You will need to be in parametric mode. Push 

and use your arrow keys to arrow down to FUNC PAR POL SEQ. Then arrow over to PAR and push 

to select it. The calculator will now be in parametric mode. Exit this screen by pushing 

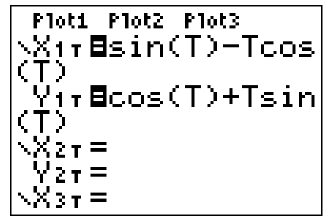
.



Enter the equations by pushing 

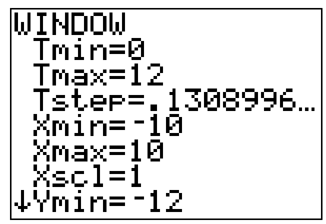
. The variable button 

will now be T. You will need to enter the *x* equation in X1 and the *y* equation in Y1. Both equations must be entered before the parametric curve can be graphed.



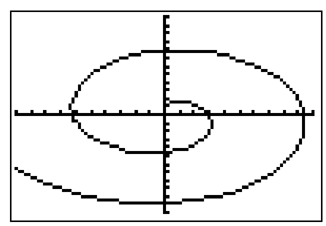
Next you will need to set your window. Push 

. The parameter T should have a minimum of 0 and a maximum of 12. You will have to determine the window that you like for the x and the y, but xmin at -10 and xmax at 10 and ymin at -12 and ymax at 12 will give you a nice view of the curve.



Graph the equation by pushing 

. Notice the orientation of the curve as it is graphed.



The graph in Example 1 is a parabola that opens to the right. This is a curve that we could have graphed without being in parametric mode. It is sometimes possible to eliminate the parameter and write an equivalent equation that is in terms of *x* and *y*. However, you may need to change the domain of the rectangular equation so that it is consistent with the domain of the parametric equation.

Example 3:
Eliminate the parameter of the curve and identify the graph of the parametric curve.

$$x = \sqrt{t},\ \ y = 2t + 1,\ \ 0 < t < 9$$

$x = \sqrt{t}$ $x^2 = t$	Solve the *x* equation for *t*.

$y = 2t + 1$ $y = 2\left(x^2\right) + 1$ $y = 2x^2 + 1$	Substitute the value of *t* in to the *y* equation.
The graph is the right side of a parabola that opens up. It has domain $[0,3]$.	

Example 4:
Eliminate the parameter of the curve and identify the graph of the parametric curve.

$$x = 3\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$$

$x = 3\cos t \qquad y = 2\sin t$ $\frac{x}{3} = \cos t \qquad \frac{y}{2} = \sin t$	Use the identity $\sin^2 t + \cos^2 t = 1$ to eliminate the parameter. First, isolate the trigonometric term in each equation.
$\left(\frac{x}{3}\right)^2 = \cos^2 t \qquad \left(\frac{y}{2}\right)^2 = \sin^2 t$ $\frac{x^2}{9} = \cos^2 t \qquad \frac{y^2}{4} = \sin^2 t$	Square both sides of each equation.
$\frac{x^2}{9} = \cos^2 t$ $+\frac{y^2}{4} = \sin^2 t$ $\frac{x^2}{9} + \frac{y^2}{4} = \cos^2 t + \sin^2 t$ $\frac{x^2}{9} + \frac{y^2}{4} = 1$	Add the two new equations together and simplify.
The equation is an ellipse centered at $(0,0)$. The domain is $[-3,3]$.	

Finding Parametric Equations

Finding Parametric Equations for a Function Given a function $y = f(x)$, the parametric equations $x = t$ and $y = f(t)$, where *t* is in the domain of *f*, can be used to define the plane curve. It is important to know that there are infinitely many pairs of parametric equations that can represent a single plane curve. The *x* equation can equal anything that will allow it to take on all the values in the domain of the original function.

Example 5:
Find three different sets of parametric equations for the function whose equation is $f(x) = x^2 - 3$.

Set 1: $x = t$ $y = t^2 - 3$	Let *x* equal *t*.
Set 2: $x = t - 1$ $y = (t-1)^2 - 3$ $y = t^2 - 2t + 1 - 3$ $y = t^2 - 2t - 2$	Let *x* equal $t - 1$.
Set 3: $x = t^3$ $y = (t^3)^2 - 3$ $y = t^6 - 3$	Let *x* equal t^3.

Applications of Parametric Equations

Projectile Motion

The parametric equations $x = (v_0 \cos\theta)t$ and $y = (v_0 \sin\theta)t + k - 16t^2$ represent the horizontal and vertical distances, in feet, of a projectile that has been launched *k* feet above ground at an angle θ to the horizontal and an initial velocity of v_0 feet per second. The parameter *t* represents the time in seconds.

Example 6:
A baseball was hit with an initial velocity of 160 feet per second at an angle of $38°$ to the horizontal. The ball was hit at a height of 3 feet above the ground.

a. Find the parametric equations that describe the position of the ball.
b. When does the ball hit the ground?
c. How far from its starting point does it land?
d. What is the maximum height of the ball during its flight?

a. $x = (v_0 \cos\theta)t$ $x = (160\cos 38°)t$ $y = (v_0 \sin\theta)t + k - 16t^2$ $y = (160\sin 38°)t + 3 - 16t^2$	Use $v_0 = 160$, $\theta = 38°$, $k = 3$, $x = (v_0 \cos\theta)t$, and $y = (v_0 \sin\theta)t + k - 16t^2$.

b. $0=(160\sin 38°)t+3-16t^2$ $t=\dfrac{-160\sin 38°\pm\sqrt{(160\sin 38°)^2-4(-16)(3)}}{2(-16)}$ $t\approx -0.030$ or 6.187 The ball will hit the ground after 6.187 seconds.	The vertical height is equal to zero when the ball hits the ground. Set the *y* equation equal to zero and solve for *t*. (Make sure that your calculator is in degrees.) Use the positive value of *t*.
c. $x=(160\cos 38°)t$ $x=(160\cos 38°)(6.187)$ $x\approx 780.1$ The ball will land about 780.1 feet from where it was hit.	Evaluate the *x* equation at the time found in part b.
d. $t=-\dfrac{160\sin 38°}{2(-16)}\approx 3.078$ seconds $y=(160\sin 38°)t+3-16t^2$ $y=(160\sin 38°)(3.078)+3-16(3.078)^2$ $y\approx 154.6$ The ball will reach a maximum height of about 154.6 feet.	Use the *y* equation. The maximum height will be reached when $t=-\dfrac{b}{2a}$. Evaluate the *y* equation at this time.

Practice Exercises A

Graph the curve defined parametrically. Be sure to indicate the orientation.

1. $x=t+2,\ y=t^2;\ -2\le t\le 2$
2. $x=t-2\ \ y=2t+1;\ -2\le t\le 3$
3. $x=t+1,\ y=\sqrt{t};\ t\ge 0$
4. $x=\sqrt{t},\ y=t-1;\ t\ge 0$
5. $x=2\cos t,\ y=2\sin t;\ 0\le t<2\pi$
6. $x=-3\cos t,\ y=5\sin t;\ 0\le t<2\pi$
7. $x=-1+\cos t,\ y=3+\sin t;\ 0\le t<2\pi$
8. $x=2t,\ y=\lvert t-1\rvert;\ -\infty<t<\infty$
9. $x=t^2,\ y=t^3;\ -\infty<t<\infty$
10. $x=t^2+1,\ y=t^3-1;\ -\infty<t<\infty$

Eliminate the parameter and identify the graph of the parametric curve. If no interval is indicated, assume that $-\infty < t < \infty$.

11. $x = 1 + t,\ y = t$

12. $x = t,\ y = t^2 - 3$

13. $x = t,\ y = t^3 - 2t + 3$

14. $x = 2t - 3,\ y = 9 - 4t;\ 3 \le t \le 5$

15. $x = 2t - 4,\ y = 4t^2$

16. $x = 2\sin t,\ y = 2\cos t;\ 0 \le t < 2\pi$

17. $x = 3\cos t,\ y = 5\sin t;\ 0 \le t < 2\pi$

18. $x = 1 + 2\cos t,\ y = 2 + 3\sin t;\ 0 \le t < 2\pi$

19. $x = 1 + 3\cos t,\ y = -1 + 2\sin t;\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

20. $x = \sqrt{t} + 2,\ y = \sqrt{t} - 2;\ t \ge 0$

21. An arrow leaves a compound bow with an initial velocity of 270 feet per second. The arrow is shot from 4 feet above the ground at an angle of $48°$ with the horizontal. Find the parametric equations that describe the position of the arrow. When does the arrow hit the ground? How far from its starting point does the arrow land?

22. A golfer at a driving range stands on a platform 12 inches above the ground and hits the ball with an initial velocity of 205 feet per second at an angle of $35°$ with the horizontal. There is a 32-foot-high fence 1200 feet away. Write the parametric equations that describe the path of golf ball. Will the ball fall short, hit the fence, or go over it?

23. A football kicked from the ground at an angle of $36°$ has an initial velocity of 65 feet per second. The goal post is 10 feet high and the ball is 105 feet from the goal post when it is kicked. Write the parametric equations that describe the path of the football. Will the football make it over the goal post?

24. A skeet is fired from the ground with an initial velocity of 112 feet per second at an angle of $25°$. Write the parametric equations that describe the path of the skeet. How long is the skeet in the air? How high does it go?

25. A baseball was hit with an initial velocity of 150 feet per second at an angle of $35°$ with the horizontal. The ball was hit at a height of 3 feet above the ground. Find the parametric equations that describe the position of the baseball. How long is the ball in flight? What is the total horizontal distance that it travels?

Geometry

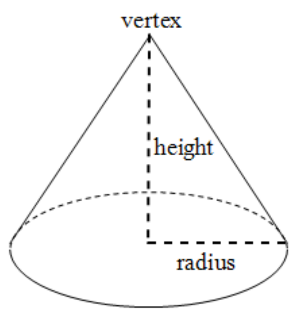
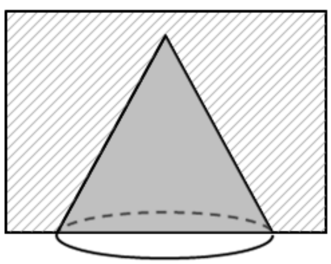
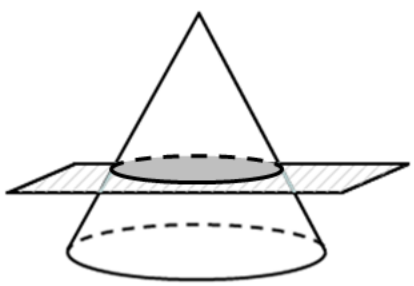
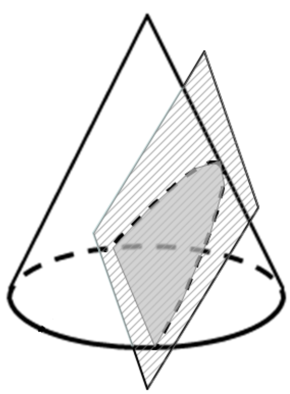
Unit 4 Cluster 7 (G.GMD.4): Two and Three-Dimensional Objects

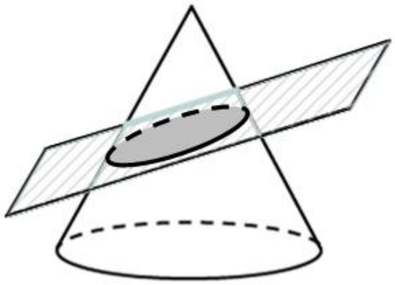
Cluster 7: Visualize relationships between two-dimensional and three-dimensional objects

4.7 Identify the two-dimensional shapes created from the cross-sections of three-dimensional objects.

4.7 Rotate two-dimensional objects and identify the three-dimensional objects created by the rotation.

Slicing or cutting through a three-dimensional figure with a plane can create a two-dimensional shape. For instance, slicing through a cone can create a triangle, circle, parabola, or ellipse.

	A **cone** is a three-dimensional figure that has a circle base and a vertex that is not in the same plane as the base. The height of the cone is the perpendicular distance between the vertex and the base.
	Slicing a cone through the vertex creates a triangle.
	Slicing a cone parallel to the base creates a circle.
	Slicing a cone diagonally, through the base, creates a parabola.

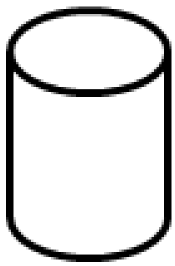
	Slicing a cone diagonally creates an ellipse.

Whenever a slice is made parallel to the base of the three-dimensional object then the two-dimensional cross-section created will be similar to the base. Additionally, the maximum number of sides that a two-dimensional cross-section can have is equal to the number of faces of the three-dimensional figure from which it is sliced.

Resources:
http://learnzillion.com/lessons/3314-visualize-crosssections-of-cones (video)
http://learnzillion.com/lessons/3386-visualize-crosssections-of-pyramids (video)
http://learnzillion.com/lessons/3469-visualize-crosssections-of-prisms (video)
http://learnzillion.com/lessons/3445-visualize-crosssections-of-cylinders (video)
http://www.learner.org/courses/learningmath/geometry/session9/part_c/ (applet)

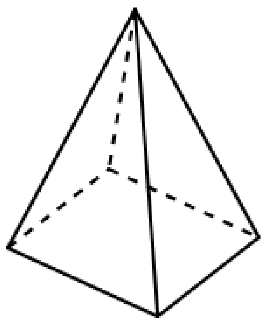
Practice Exercises A

Determine the two-dimensional cross-section that is created from each slice described.



Right Cylinder

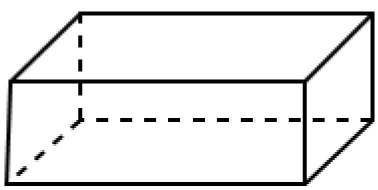
1. Horizontal slice.
2. Vertical slice.
3. Diagonal slice (not through a base).



Square Based Pyramid

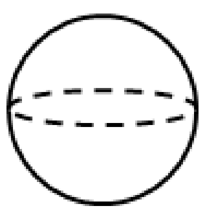
4. Horizontal slice.
5. Vertical slice through the vertex opposite the base.
6. Vertical slice not through the vertex opposite the base.
7. Diagonal slice through all four lateral sides and the base.

Right Rectangular Prism



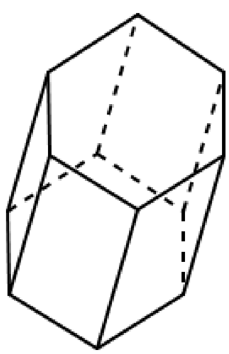
8. Horizontal slice.
9. Vertical slice.
10. A slice that cuts off a corner.
11. Diagonal slice through one base and all of the lateral sides.

Sphere



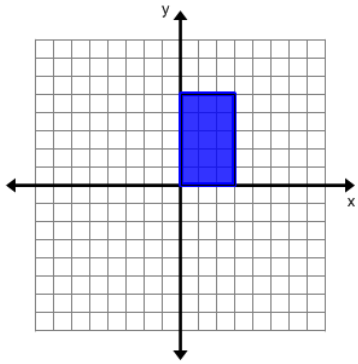
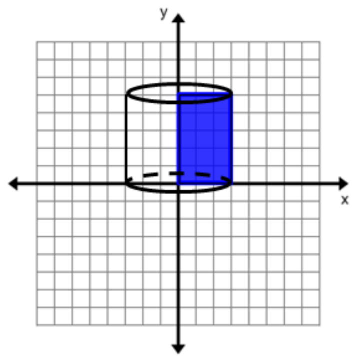
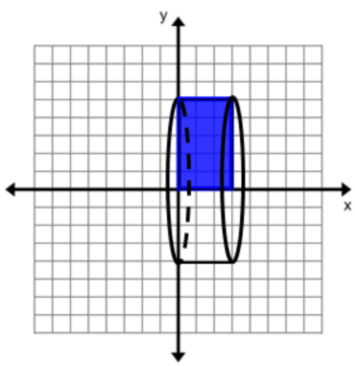
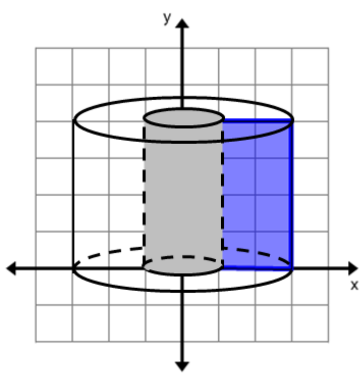
12. Horizontal slice.
13. Vertical slice.
14. Diagonal slice.

Hexagonal Prism



15. Horizontal slice.
16. Vertical slice.
17. A diagonal slice through all the lateral faces and one of the bases.
18. Can you make an octagon by slicing the shape? Why or why not?

Rotating a two-dimensional figure around an axis creates a three dimensional figure.

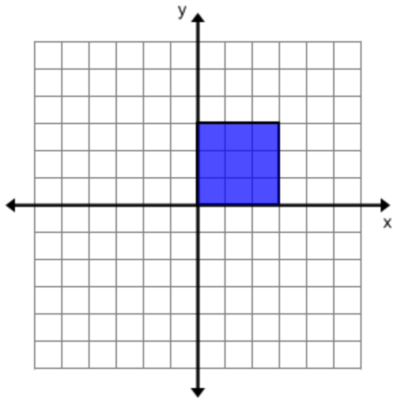
	Start with a rectangle that has a side on the each axis.
	Rotating around the *y*-axis creates a right circular cylinder with a height *y* and radius *x*.
	Rotating around the *x*-axis creates a right circular cylinder with a height *x* and radius *y*. Notice that the side perpendicular to the axis of rotation is flat, while the side parallel is curved.
	Rotating a rectangle that has only one side on an axis creates a cylinder with a hole in the middle or a doughnut.

Resources:
http://learnzillion.com/lessons/3488-predict-3d-results-of-rotating-simple-figures (video)

Practice Exercises B

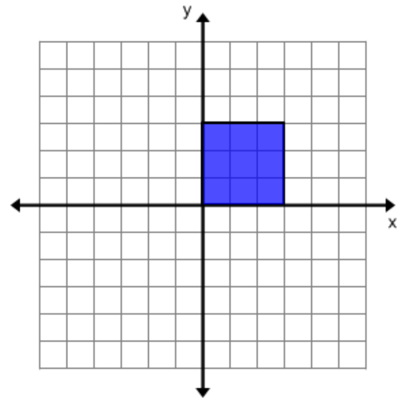
Sketch the result of rotating each shape around the given axis.

1.



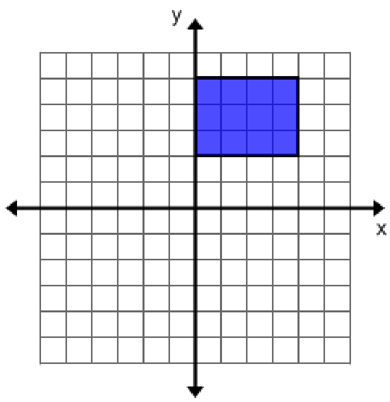
Around the y-axis

2.



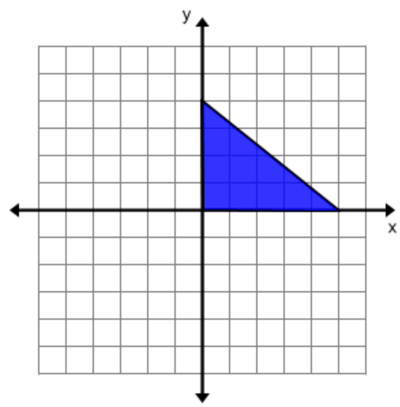
Around the x-axis

3.



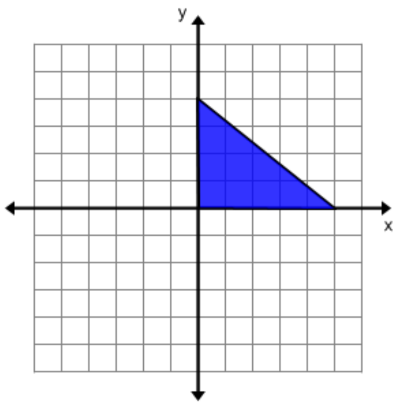
Around the x-axis

4.



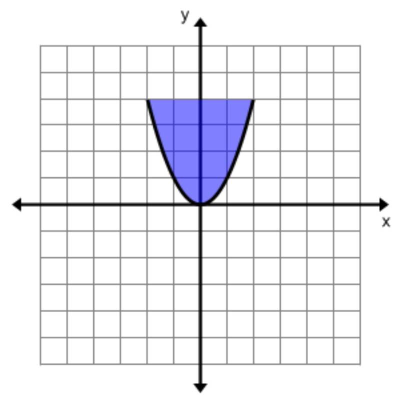
Around the y-axis

5.



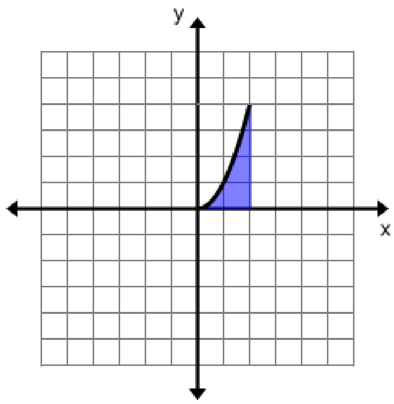
Around the x-axis

6.



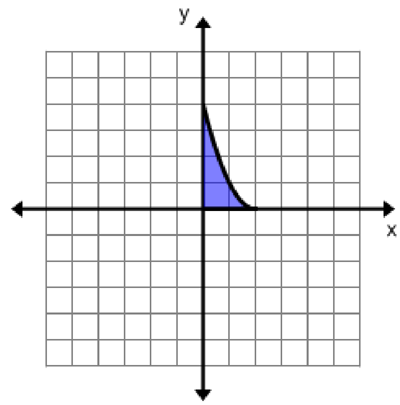
Around the y-axis

7.



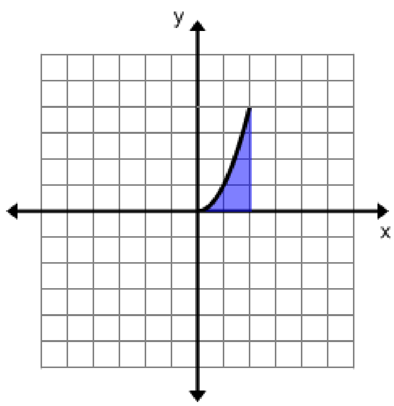
Around the *x*-axis

8.



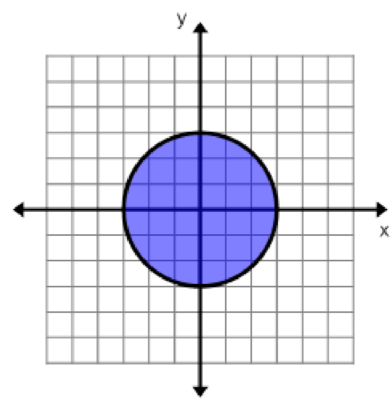
Around the *y*-axis

9.



Around the *y*-axis

10.



Around the *y*-axis

Unit 4 Cluster 8: G.MG.1, G.MG.2, & G.MG.3 Mathematical Modeling

Cluster 8: Apply geometric concepts in modeling situations

4.8 Use geometric shapes, their measures, and their properties to describe objects.

4.8 Apply geometric methods to solve design problem.

Using geometric shapes, you can estimate the areas and volumes of complex everyday objects.

Example 1:

How much surface area is painted on a regular yellow #2 pencil?

Looking at the end of a pencil before you sharpen it, you can see that a pencil is actually a hexagonal prism. Finding the lateral surface of the pencil will tell you how much of the surface is painted.	
$s = 4$ mm $l = 17.1$ cm	Find the length of one side of the hexagonal base. And the length of the pencil not including the eraser.
$s = 4$ mm $l = 171$ mm	Change each length to the same units of measure.
Painted Area $= (6)(4)(171)$ $= 4{,}104\text{ mm}^2$	There are six (6) lateral faces shaped like rectangles. Dimensions $4\text{ mm} \times 171\text{ mm}$

Example 2:

You are redecorating your house and want to paint an accent wall in your dining room. You will use a paint roller to paint the wall.

a. How much area will one rotation of the roller cover?

b. How many rotations will it take to cover a $8' \times 10'$ wall?

a. $C = \pi d$ $C = \pi(2.5)$ $C \approx 7.854$ inches	The diameter of a paint roller is 2.5 inches. The length is 9 inches. Find the circumference of the paint roller.
$A = Cl$ $A \approx (7.854)(9)$ $A \approx 70.686\text{ in}^2$ One rotation of the roller covers 70.686 in^2	Find the area that one rotation of the roller will cover in paint.

b. $$(8)(12)=96\text{ inches}$$ $$(10)(12)=120\text{ inches}$$ The wall is $96''\times 120''$	Change the dimensions of the wall to inches.
$$A=(96)(120)$$ $$A=11{,}520\text{ in}^2$$	Find the area of the wall.
Number of rotations $\approx \dfrac{11520}{70.686}$ ≈ 162.975 You would need 163 full rotations to cover the wall with paint.	Divide the area of the wall by the area of the roller.

Example 3:
You need to buy bark to finish the landscaping of your yard. You have four flower beds measuring $90'\times 12'$, $3'\times 15'$, $10'\times 4'$, and $1.5\times 22'$. The bark needs to be 6 inches deep. You can buy it by the bag for \$6.25 which covers 2 cubic feet or in bulk for \$60 which covers 1 cubic yard. Which is the least expensive way to purchase your bark?

$A=lw$ $A_1=(90)(12)$ $=1{,}080\text{ ft}^2$ $A_2=(3)(15)$ $=45\text{ ft}^2$ $A_3=(10)(4)$ $=40\text{ ft}^2$ $A_4=(1.5)(22)$ $=33\text{ ft}^2$	Find the area of each flower bed.
$A_1+A_2+A_3+A_4=1080+45+40+33$ $=1{,}198\text{ ft}^2$	Find the total area of the flower beds.
6 inches =0.5 feet Volume $= Area(depth)$ $=(1198)(0.5)$ $=599\text{ ft}^3$	Find the volume of bark you need, but first convert the depth to feet.
$\dfrac{599}{2}\approx 299.5$ You will need 300 2-cubic foot bags of bark	How many 2-cubic foot bags would you need? Divide the volume by 2.
$(300)(6.25)=1{,}875$ It would cost \$1,875 to buy 300 2-cubic foot bags of bark	Find the cost of buying 300 bags. Multiply the number of bags by \$6.25
1 cubic yard $=(3)(3)(3)$ $=27$ cubic feet	Convert 1 cubic yard to cubic feet

$\frac{599}{27} \approx 22.185$ You would need 23 loads of bulk material	Divide the volume of bark you need by 27.
$(23)(60)=1380$ It would cost $1,380 to buy 23 loads of bulk bark	Find the cost of buying 23 loads of bulk bark. Multiply the number of loads by $60
You would save $495 by buying in bulk instead of in bags.	

Practice Exercises A

1. The base of a water bottle has diameter of 3.5 inches. The label wraps around the entire base of the bottle and has a height of 3.25 inches. What is the approximate surface area of the label?

2. You have a bookcase that is 78 inches high and 28 inches wide. It has 6 equally spaced shelves. You have textbooks that are $10\frac{7}{8}$ inches long and $1\frac{1}{8}$ inches wide. What is the maximum number of textbooks that can be stored in the bookcase?

3. You have decided to add a door to an existing wall. The wall is $10' \times 25'$. The door will be $3' \times 7'$. You want to finish the wall with brick and the size of a standard brick is $9'' \times 3''$. How many bricks will you need to complete the wall?

4. You are going to retile your shower. The dimensions of the shower are $32'' \times 32'' \times 73''$. You will be tiling 3 walls, the floor and the ceiling. It has a glass door but you will have to tile $6''$ below the door. You have three choices of tile: $12'' \times 12''$ square tile for $1.88 per square foot, $6'' \times 6''$ square tile for $3.16 per square foot, and $2'' \times 2''$ square tile for $3.65 per square foot. If you want to minimize the cost, which tile should you use, how many tiles will you need, and how much will it cost?

Challenge

You have been asked to design a parking lot for a new community center. It must have 200 parking stalls, 6 of which must be designated handicapped parking. All adjacent rows of stalls are perpendicular to each other. Regular stalls are 9 feet wide and 18 feet long. Handicapped stalls must be 8 feet wide and 18 feet long with a 5 foot access aisle on one side. Your parking lot needs 24 feet between parking rows. Keep in mind you need to minimize the square footage used for you parking lot.

VOCABULARY Density describes the amount of matter per volume of an object. Density can be compared by using the weight of a standard amount of the substance. The amount most scientists use is the cubic centimeter $\left(\text{cm}^3\right)$. For example, a cm^3 of helium weighs 0.0001785 grams, while cm^3 of air weighs 0.001205 grams. The helium is lighter than air which is why helium floats! Density can also describe the number of items in an area per square unit. For example population density measures the number of people in a given area.

Example 4:

You have a solid gold half sphere paper weight. It has a volume of approximately 17 cm^3. You want to replace it with a similar paper weight filled with sand. If the density of gold is about 19.32 g/cm^3 and the density of sand is about 2.5 g/cm^3, will the two paper weights weigh the same?

Gold Paper Weight: $19.32 = \frac{\text{Mass}}{17}$ $(17)(19.32) = \text{Mass}$ $328.44 \text{ grams} = \text{Mass}$ Sand Paper Weight: $2.5 = \frac{\text{Mass}}{17}$ $(17)(2.5) = \text{Mass}$ $42.5 \text{ grams} = \text{Mass}$ The two paper weights will not weigh the same.	Find the weight of each paper weight using $\text{Density} = \frac{\text{Mass}}{\text{Volume}}$ Solve for Mass

Example 5:

The 2010 census indicated that Utah had a population of 2,763,885. The area of Utah is 84,899 square miles. Find the population density of Utah.

$\text{Density} = \frac{\text{Population}}{\text{Area}}$	Use the density formula.
$\text{Density} = \frac{2{,}763{,}885}{84{,}899}$ $\approx 32.555 \text{ people per square mile}$	Substitute known values for the population and area. Divide.

Practice Exercises B

1. Texas has an area of 268,581 square miles and a population of 26,059,203 people. Vermont has an area of 9,614 square miles and a population of 626,011. Which state has the highest population density?

2. Strawberry Reservoir has a water surface of 26.817 square miles and Fish Lake has a water surface of 3.907 square miles. How many fish would you have to stock in both bodies of water in order for their fish population densities to be 500 fish per square mile?

3. You have a $2'' \times 4'' \times 8'$ cedar plank which weighs approximately 10.2 pounds and a $1'' \times 2'' \times 10'$ Oregon pine board which weighs approximately 4.6 pounds. Find the density of each piece of lumber and decide which wood is denser.

Statistics

Unit 1 Cluster 2 (S.IC.1): Statistical Inferences

Cluster 2: Understand and evaluate random processes underlying statistical experiments

1.2 Understand that statistics allows inferences to be made about population parameters based on a random sample from that population.

Suppose someone wanted to do a study about all sophomores, juniors, and seniors currently in high school in the United States. They might want to know the number of students who work after school, own a cell phone, or participate in extracurricular activities. Through statistical methods, we can gather and analyze information from a smaller sample population which allows us to make inferences about the much larger entire population. In Jordan School District there are currently 11,376 sophomores, juniors, and seniors in high school. It would be extremely costly and time consuming to interview every student; however, by taking a random sample of students we can calculate statistics which will allow us to draw conclusions about all 11,376 students in high school in Jordan School District.

VOCABULARY

A **population** consists of all people or items which we wish to describe or draw conclusions about. A **sample** is a small group of people or items taken from the larger population.

The population characteristic that we are interested in is called the **parameter of interest**. In the case of our high school example, the parameter of interest could be the number of students who work after school.

If it were possible to gather data from an entire population, then the parameter of interest would be the **population parameter**.

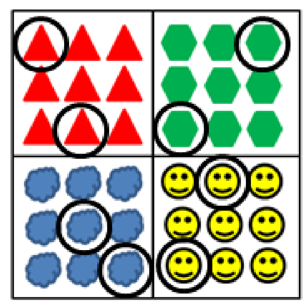
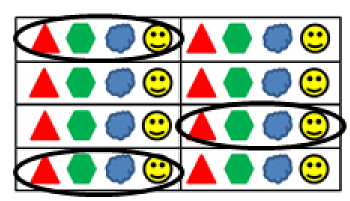
It is often difficult to gather data from an entire population so we use **statistics**, or data that we gather from a sample of the population, to make an **inference** or conclusion about the parameter of interest for the population.

In obtaining a sample from a population it is important to use **random sampling** to ensure the sample is representative of the population. Random sampling is a technique where a group is selected from the population. Each individual is chosen entirely by chance and each member of the population must have an equal chance of being included in the sample.

Example 1:
The Utah State Legislature wants to know what percentage of teen drivers text while they drive. They decide to survey 250 randomly selected teen drivers across the state. Identify a) the population, b) the sample population, and c) the parameter of interest.

a) the population is all teen drivers in the state of Utah
b) the sample is 250 teen drivers in the state of Utah
c) the parameter of interest is the percentage of teen drivers that text while they drive

Obtaining a random sample is not as simple as you might think. In fact, there are a few different methods for sampling. Some of the methods can be biased. A bias occurs when part of the population is overrepresented or underrepresented. For example, if you wanted to know how many students support the school's athletic programs, you wouldn't interview only the cheerleaders or students on a team, because they regularly attend athletic events and would be overrepresented in the study.

Sampling Methods	
In a **simple random sample** every member of the population has an equal chance of being selected to be part of the sample group. Drawing names from a hat is an example of this type of sampling. Another example would be assigning every member of the population a number and then using a random number table or generating random numbers through technology to randomly select members. The key is that you must have a list of all the members of the population.	
In a **systematic sample** it is assumed that the entire population is naturally organized in a sequential order. Using a random number generator, you select a starting point and then select every nth member to be part of the sample. For example, homes in a neighborhood are already in an order. You could randomly select a starting point and then select every third home.	
In a **stratified sample** members of the population that share the same characteristic are grouped together. Then, members of that subgroup are randomly selected to make up the sample group. Each member of the subgroup has an equal chance of being selected. There are times when the subgroups are not equal in size. When this happens, members are chosen in proportion to their actual percentages in the overall population. For example, if you wanted to study all high school students who are involved in extracurricular activities you would probably want to divide them into their particular extracurricular activity and then select randomly from those groups so that each extracurricular activity is represented in the sample. The football team would have more members than the basketball team so you would select more football players than basketball players to participate.	
In a **cluster sample** the population is divided into smaller groups that are representative of the entire population and then groups are randomly selected. For example, if you wanted to make inferences about your entire school, you could randomly select 1^{st} periods to survey.	
In a **convenience sample** members are randomly selected from a population that is readily available. For example if you wanted to ask shoppers what they think of a local store, you would survey every 5^{th} person who exits the store on a given day. This method of sampling has a bias because people who like to shop at this particular store are more likely to be at the store that day.	
In a **volunteer sample** members of the population self-select to be included in the sample. Filling out a survey and returning it is an example of a volunteer sample. This is prone to bias because generally people who respond have strong opinions about the topic while others who are more neutral may not respond at all. An example of a volunteer sample is when you buy a pair of shoes at your favorite shoe store and the cashier asks you to complete an online survey about your experience that day. You decide whether or not you want to complete the survey.	

Example 2:
The school newspaper wants to know the percentage of students who drive to school each day. For each method described below, determine what type of sampling method it is and justify whether or not the method is biased.

a. The newspaper staff posts signs all over the school asking students to take a short survey online.
b. The newspaper staff interviews every fifth person who walks into the school cafeteria.
c. The newspaper staff randomly selects 20 fifth periods to survey.

a.	Volunteer Sample. This may or not be biased depending upon the question, how lengthy the survey is, and how difficult it is to complete.
b.	Convenience Sample. This is biased because students who have cars are more likely to leave campus for lunch.
c.	Cluster Sample. This is generally not biased.

Example 3:
You want to know if students at your school prefer fast food or sit-down restaurants. What would your survey question look like to eliminate any bias? Explain the sampling method you would use and why?

There are many sampling methods that would be appropriate and unbiased. This is one example.

The question: Where do you like to eat out?

Method: Systematic Sampling. The entire school population is generally known. I can obtain a list of the students in alphabetical order. I can randomly select one of the first ten students and then select every tenth student from there on. This eliminates bias because every student has an equal chance of being selected.

Practice Exercises A

For each situation described, identify a) the population, b) the sample population, and c) the parameter of interest.

1. An AP Government class wants to know the percentage of eligible voters in the state of Utah who voted in the most recent election. There are 1,938,249 people in Utah who are 18 and older. The class randomly looks at 15 state house districts and discovers that 50.5% of the eligible voters actually voted.

2. A local radio station has added an additional radio personality and is trying to determine what type of music to play during this person's air time. This time slot is geared towards teenage listeners. The station has decided to survey 300 randomly selected students from the ages of 13 to 19.

3. A health class wants to know the average amount of time Utahns over the age of 12 spend exercising each week. A sample of 1,200 randomly selected people, over the age of 12, across the state was surveyed.

For each method described below, determine what type of sampling method it is and justify whether or not the method is biased.

4. In order to determine the average composite score on the most recent ACT exam, students were divided into groups based on whether they were enrolled in remedial, regular, or honors language arts. Individual scores were randomly selected from each group.

5. Every third patron exiting the school musical was surveyed regarding their support for more funding of the arts.

6. A random number generator was used to assign students to demonstrate work in front of the class.

Project

7. Find a question of interest about the school population. Collect a random sample about the question of interest. Determine what inferences can be made about the population from that sample.

Unit 1 Cluster 2 (S.IC.2): Simulation

Cluster 2: Understand and evaluate random processes underlying statistical experiments
1.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.

VOCABULARY

The **theoretical probability** of an event occurring is the ratio of the number of favorable outcomes to the total number of outcomes, $P(\text{event}) = \dfrac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$. It is what should happen in theory. For example, if we roll a die (a six-sided number cube) 60 times, in theory, the results should be 10 ones, 10 twos, 10 threes, etc. However, if we were to roll a die 60 times and calculate the probability based upon our data, the results may differ. The probability that we calculate from data we collect is the **experimental probability** of an event. It is defined to be the ratio of the number of favorable outcomes to the number of trials, $P(\text{event}) = \dfrac{\text{number of favorable outcomes}}{\text{number of trials}}$. The **law of large numbers** says that the more trials you run in an experiment, the closer the experimental probability will get to the theoretical probability.

Simulations can use any random method of generating results as long as it fits the situation. A coin can be used to simulate an event with only two outcomes. A random number table can be used to simulate an event with 10 or 100 outcomes. A deck of cards can be used to simulate an event with 4, 13, or 52 outcomes. A random number generator, like a calculator, can be used to simulate any number of outcomes.

Example 1:
The school newspaper staff says it randomly picked 20 students to interview about the school's dress code. Fifteen of the students selected were boys. Does the number of boys selected cause you to question the selection process?

There are several simulations that you could use to model this situation.

a. **flipping a coin:** Since there are only two outcomes: boys or girls, this is an acceptable method for simulating the selection. Let heads = boys and tails = girls. Flip a coin 20 times and record your data. Each set of 20 flips represents a single trial. Several trials are necessary in order for the simulation's statistic to be a good approximation.

Trial	Heads Boys	Tails Girls
1	9	11
2	12	8
3	11	9
4	8	12
5	13	7
6	6	14
7	7	13
8	10	10

9	13	7
10	9	11
11	7	13
12	13	7
13	12	8
14	13	7
15	9	11

From the 15 trials you can see that 15 heads and 5 tails never occurred. It is highly unlikely that the selection was random.

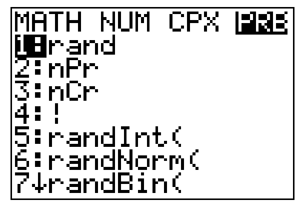
b. **using a random number generator:** the TI-83 and TI-84 calculators have a random number generator feature.

The random integer feature is in the MATH menu. Push 

then use your arrow keys to arrow over to PRB. Option number 5 randInt(is the feature you want to use. Select it by pushing 

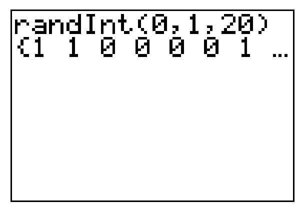
or using your arrow keys to arrow down to 5 and pushing 

.



The syntax is randInt(lower number, upper number, number of numbers you want returned). In our simulation we can use 0 to represent boys and 1 to represent girls. Twenty students were chosen so we want to have 20 numbers returned. Use your arrow keys to see the numbers that are off the screen. By pushing 

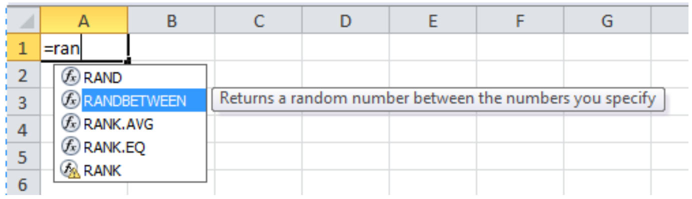
repeatedly you can do several trials.



Trial	**0 Boys**	**1 Girls**
1	9	11
2	10	10
3	12	8
4	8	12
5	7	13
6	13	7
7	13	7
8	8	12
9	9	11
10	8	12
11	14	6
12	10	10
13	14	6
14	8	12
15	10	10

The data is consistent with flipping a coin. There were no occurrences of 15 and 5.

Microsoft Excel also has a random number generator feature that is similar to the calculator's random number feature.



The syntax is =RANDBETWEEN(lower number, upper number). Let 0 = boys and 1 = girls then in a cell type =RANDBETWEEN(0,1). You can copy the formula down the column so that it is in 20 cells. Copy the column several times and you have your simulation.

Trial	**0 Boys**	**1 Girls**
1	9	11
2	9	11
3	10	10
4	12	8
5	10	10
6	12	8
7	9	11
8	9	11
9	11	9
10	8	12
11	7	13
12	9	11
13	10	10
14	9	11
15	8	12

The data is consistent with flipping a coin and the calculator's random number generator. There were no occurrences of 15 and 5.

c. **random number table**: random number tables can be found on the internet. This particular table uses numbers 0 to 9 and groups them in clusters of 5. Let the even numbers including zero represent boys and the odd numbers represent girls. Use a random number generator to select a row to begin at. In this simulation, row 7 is where we will begin.

Random Number Table

16291	47751	28617	43266	75692	81384	25354	78664	35358	14658
93761	93658	15455	18589	64916	51584	17368	37478	53769	62767
76772	24458	49349	26977	55973	94643	77369	44195	68696	44356
64883	45331	43386	94778	35279	46898	63253	81918	63219	57955
64686	99491	32921	21687	27593	89286	56643	81317	94334	35217
67123	54977	86575	42722	91337	84614	76229	67517	23953	43454
39787	57814	17496	37277	43156	21483	44215	69351	11536	51665
87251	52193	94179	65383	26512	16476	56585	85955	25919	65346
51437	78564	57291	99419	15222	64582	62473	25812	26869	41256
69143	31827	31237	55455	47444	87593	97638	57597	68126	59583
72828	24116	42381	25452	14434	15131	53789	55711	75147	96269
86675	68946	62963	58266	54867	23988	97653	34312	31265	15965
46672	78525	64155	29222	47717	93568	65534	17878	97237	85737
24575	34765	61588	335411	57237	64314	51587	28797	46111	81988
42941	71328	39677	27853	25119	65448	84123	55469	46175	44911

The first group of numbers is: 3, 9, 7, 8, 7, 5, 7, 8, 1, 4, 1, 7, 4, 9, 6, 3, 7, 2, 7, and 7. There are 6 that are even or zero and 14 that are odd.

Trial	**Even Boys**	**Odd Girls**
1	6	14
2	9	11
3	6	14
4	9	11
5	7	13
6	6	14
7	13	7
8	11	9
9	7	13
10	7	13
11	13	7
12	4	16
13	11	9
14	13	7
15	6	14

The data is consistent with the other methods for simulation. There were no occurrences of 15 and 5.

There are many different ways to simulate a model. A die, colored marbles, or colored chips could also be used for simulations. You just need to make sure that the simulation is random and representative of the situation.

Practice Exercises A

Run a simulation of 50 trials to determine the probability indicated.

1. Given a 85% attendance rate, what is the probability that out of a class of 40 students exactly 34 will be present on any given day?

2. Lindsey has a 78% free-throw shooting record. What is the probability of having a 64% free-throw record in any given game?

3. Assume that 50% of the students enrolled in AP Calculus classes are male. In Ms. J’s class, there are 24 students, 8 of which are female. What is the probability that this would happen by chance?

Unit 1 Cluster 3 (S.IC.3 & S.IC.6): Surveys, Experiments, Observations, and Evaluation of Reports

Cluster 3: Make inferences and justify conclusions from sample surveys, experiments, and observational studies

1.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

1.3 Evaluate reports based on data.

There are three main techniques for gathering a sample: sample surveys, experiments, and observational studies. Each of these methods has a purpose, advantages, and limitations. Randomization should occur in each of these methods.

VOCABULARY

When conducting a **survey** every member in the sample answers a set of questions.

Experiments require at least two groups. One group receives the trial treatment, while the other, sometimes called the control group, does not receive the treatment. At the end of an allotted period of time, the two groups are compared to determine if the treatment had an effect.

Observational studies require you to observe outcomes without interacting with any members of the sample.

Sample Surveys

The purpose of a sample survey is to gather information about the sample by means of a survey. There are several advantages to using a survey. Surveys are inexpensive and can collect a large amount of data representative of the population. They can be done in a variety of forms and about a variety of topics. Surveys also have the ability to focus only on the necessary information. However, surveys are flawed by non-responders since a survey is generally voluntary; people have the option not to participate. Additionally, people in a survey know that they are being studied and they may not be as honest in their responses as they would be if they were not being studied. Surveys are also open to interpretation and bias. Surveys can be written in a way that biases the responders. Also questions can be interpreted differently than intended by those responding to the survey.

Surveys can be administered with randomization methods, such as simple random sampling, cluster sampling, multistage sampling, stratified sampling, or systematic sampling all of which would ensure that the sample is random and representative of the overall population.

Experiments

The purpose of an experiment is to assign a treatment, using control over some of the conditions in order to gather data about the treatment's effectiveness. An experiment is the only way to establish causation. When an experiment is designed, all of the variables are controlled. This allows the experimenter to demonstrate that a change in one variable causes the change in another variable. There are drawbacks to experiments. They can be very expensive and time consuming. Ethics may be questioned especially if animals or people are used in the experiment. Experiments must not intentionally harm any of the subjects. The attitude and behavior of those conducting the experiment can also affect the results.

It is imperative that randomization is used when assigning subjects to their treatment groups. Each group needs to be representative of the overall population.

Observational Studies

The purpose of an observational study is to observe subjects in their natural environment without their knowledge and without assigning treatments to the subjects. There are some advantages to using an observational study. It is simple and inexpensive to conduct. It provides deeper and richer information than a survey because the observer is seeing behavior firsthand and is able to observe the process not just the result. There are also some disadvantages. The results cannot prove causation nor can they be applied to the general population. It is only representative of those being studied. The results are subjective and open to interpretation by the observer. There may also be a question of ethics, especially if people are involved. People have a right to privacy and the observational study must not infringe upon the rights and expectations of people.

If you are doing the study in the present, you can randomize the individuals involved. If you are gathering data from past records, there is no chance for randomization.

Example 1:
Which type of study method is described in each situation? Should the sample statistics be used to make a general conclusion about the population?

a. Researchers randomly choose two groups from 20 volunteers. Over a period of 6 weeks, one goup works on a computer for an hour right before going to sleep, and the other does not. Volunteers wear monitoring devices while sleeping, and researchers record their quality of sleep.

b. Students in an elementary class observe the growth of some newly hatched chickens.

c. Market researchers want to know if people like the new store at the local mall. They ask every fourth person who enters the mall if they like the new store.

Answer:

a. This is an experiment. There are two groups: a treatment group and a control group. There are very few participants in the study so it is not a good idea to generalize this to the entire population.

b. This is an observational study. Observational studies can't be generalized to the entire population. There may be variables that are not controlled such as amount of food available, the climate, etc.

c. This is a survey. The results of the survey can only be applied to the population that shops at the local mall. It wouldn't account for the opionion of those who do not shop at the local mall.

Practice Exercises A

Which type of study method is described in each situation? Should the sample statistics be used to make a general conclusion about the population?

1. A numbered list of students is generated from the school database. Students are randomly selected from the list by using a random number generator. Information for every student is entered into the database, and each student has an equally likely chance of being selected. The students selected are asked how much allowance money they are given each week for doing chores at home.

2. The owner of a bakery collects data about the types of cupcakes that are purchased so she can make cupcakes accordingly. She records they type of cupcake purchased by every other person each day for three weeks.

3. A gardener tests a new plant food by planting seeds from the same package in the same soil and location. He is careful to water the plants the same, but gives one plant food and the other no food at all. He records the growth and flowering rates of each plant.

4. Every day for two weeks, a student records the number of her classmates who are late to class.

5. A local grocery store selects 350 customers from a list of 1500 new customers in the past year to mail a questionnaire. There are 245 customers who return the questionnaire.

6. A teacher wants to know if playing classical music while a class works on a test will improve their scores on the test. She uses two class periods of equal size (35 students in each class) and equal baseline test data. For an entire semester, she plays classical music while one class is testing and plays no music while the other class is testing.

Evaluate Reports

Statistics are reported everywhere. It is important to look at any statistical reporting critically and evaluate the content for its validity in regards to the general population. Here are some things to consider when evaluating any report containing statistical information.

- Sampling method
- Study type
- Population of interest
- Bias
- Sample size
- Study duration

Practice Exercise B

1. Use the criteria above to evaluate the reports found at the links below.

   Dan Jones and Associates (Utah based company)
   http://cppa.utah.edu/_documents/publications/governance/hb-40-final-report.pdf

   Gallup Student Polls (National Company)
   http://www.gallupstudentpoll.com/159221/gallup-student-poll-overall-scorecard-fall 2012.aspx

Unit 1 Cluster 1 (S.ID.4): Normal Distribution

Cluster 1: Summarize, represent, and interpret data on a single count or measurement variable

1.1 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages.

1.1 Recognize that there are data sets for which such a procedure is not appropriate.

1.1 Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

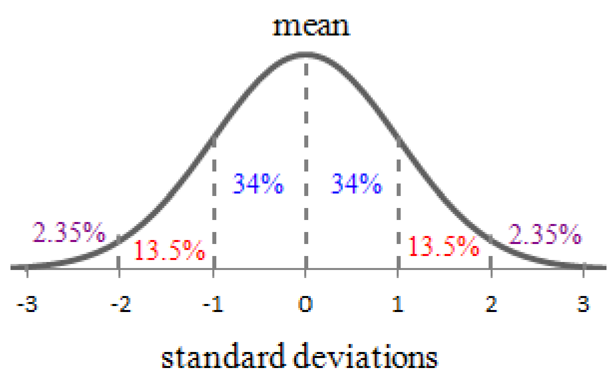
VOCABULARY

Data that is unimodal, symmetric, and with no outliers is said to be normally distributed. A **normal distribution** is bell shaped with mean, μ, at the center of the curve.

In a normal distribution,

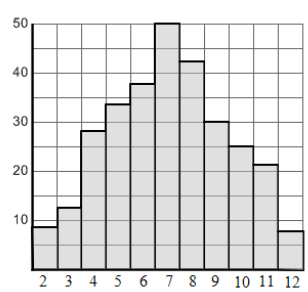
- 68% of the data fall within one standard deviation, $\pm\sigma$, of the mean
- 95% of the data fall within two standard deviations, $\pm 2\sigma$, of the mean
- 99.7% of data fall within three standard deviations, $\pm 3\sigma$, of the mean.

Nearly all data lie within three standard deviations from the mean. This is known as the **empirical rule**.



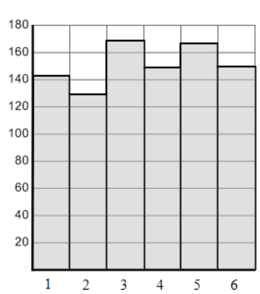
The area under a normal curve is always 1. When calculating population percentages, the value will be less than 1.

Normal Distribution

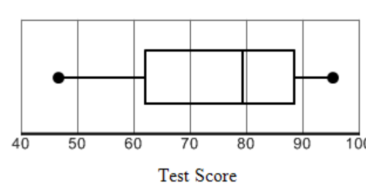


Not all data sets are distributed normally. Notice the examples below. Some data sets are uniform, left or right skewed, or bimodal.

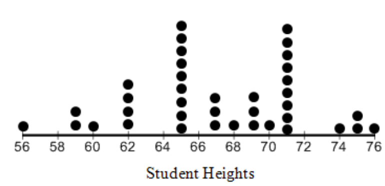
Uniform Distribution



Left Skewed



Bimodal



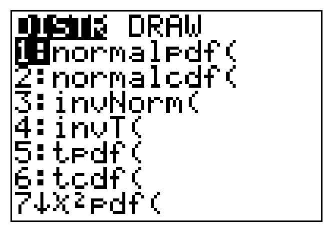
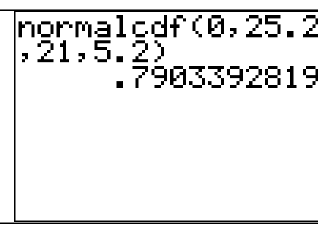
Example 1:
ACT test scores are approximately normally distributed. One year the scores had a mean of 21 and a standard deviation of 5.2.

a. What is the interval that contains 95% of scores?
b. What percentage of ACT scores is less than 25.2?
c. What percentage of ACT scores is between 28 and 36?

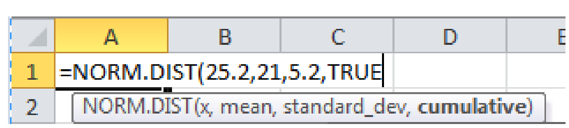
a. $\mu \pm 2\sigma$ $21 \pm 2(5.2)$ 21 ± 10.4 $21 - 10.4 = 10.6 \qquad 21 + 10.4 = 31.4$ The interval that contains 95% of the scores is from 10.6 to 31.4.	The interval containing 95% of the scores would be $\mu \pm 2\sigma$. $\mu = 21$, $\sigma = 5.2$

b. Since the score 25.2 is not quite one standard deviation away from the mean, $x - \mu = 25.2 - 21 = 4.2$, the empirical rule cannot be used to calculate the percentage. You will need to use a graphing calculator or a spreadsheet to calculate the proportion.

Using a TI-83 or TI-84 Graphing Calculator

The distribution features are found by pushing  . A menu like the one at the right should appear. Option 2, normalcdf(is the feature that you want to use. This feature is the normal cumulative distribution function. It will calculate the percentage of data that fall between two numbers. Select option 2 by pushing  or by using your arrow keys to arrow down to 2 and pushing .	
The syntax required for this feature is normalcdf(lower bound, upper bound, mean, standard deviation). In the case of our example, it would be normalcdf(0, 25.2, 21, 5.2). Approximately 79% of the scores are below 25.2.	

Using an Excel Spreadsheet

In a cell start an equation by typing an equal sign and then type NORM.DIST. The syntax for entering information is NORM.DIST(value, mean, standard deviation, cumulative). By stating that the cumulative part is TRUE the entire percentage up to the value specified will be calculated.	

In the case of our example, you should type =NORM.DIST(25.2, 21, 5.2, TRUE) and then push enter. Approximately 79% of the scores are below 25.2.	A 1 0.790366

c. Approximately 9% of the scores are between 28 and 36.	Use a calculator to find the percentage. Use normalcdf(. normalcdf(28,36, 21,5.2) .0871669871
	You could also use Excel, but you would have to do a little extra work. You would need to find the percentage that scored less than 28 and the percentage that scored less than 36 and then find the difference between the two. A 1 0.910874 2 0.998041 3 0.087167

Practice Exercises A

1. The mathematics portion of the SAT has a mean score of 500 and a standard deviation of 100.
   a. What is the interval that contains 99.7% of scores?
   b. What percentage of SAT scores is greater than 650?
   c. What percentage of SAT scores is between 325 and 615?

2. Americans consume 16.5 pounds of ice cream per year with a standard deviation of 3.25 pounds.
   a. What is the interval that contains 68% of the pounds consumed each year?
   b. What percentage of pounds consumed is less than 10 pounds?
   c. What percentage of pounds consumed is between 5 pounds and 11pounds?

3. The average height of a NBA basketball player is 79 inches with a standard deviation of 3.89 inches.
   a. What is the interval that contains 95% of the heights?
   b. What percentage of the heights is greater than 81 inches?
   c. What percentage of the heights is between 73 inches and 77 inches?

Unit 1 Cluster 3 (S.IC.4): Margin of Error

Cluster 3: Make inferences and justify conclusions from sample surveys, experiments, and observational studies

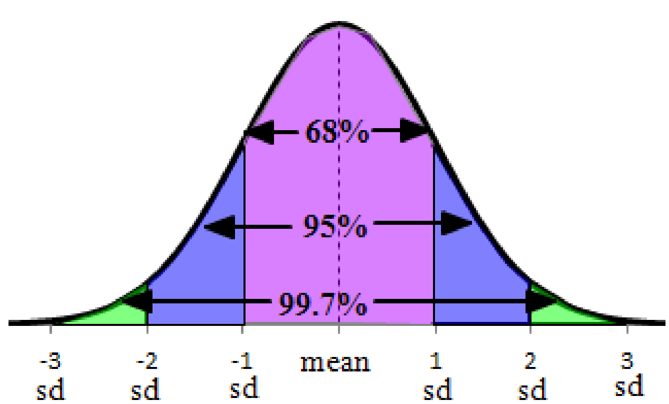
1.3 Use data from a sample survey to estimate a population mean or proportion.
1.3 Develop a margin of error through the use of simulation models for random sampling.

When we use a simulation to model an event, it is only an approximation of the population parameter. If we were to run the simulation numerous times, each result would be slightly different. However, it is possible to give an interval that the population parameter falls within by finding a margin of error.

A margin of error is not a mistake; rather it refers to the expected range of variation in a survey or simulation if it were to be conducted multiple times under the same procedures. The margin of error is based on the sample size and the confidence level desired. The interval that includes the margin of error is called the **confidence interval** and is usually computed at a 95% **confidence level**. A confidence level of 95% means that you can be 95% certain that the actual population parameter falls within the confidence interval.

For example, the student body officers at your school conducted a survey to determine whether or not a majority of the students dislike the music played in the hall during class changes. They randomly interviewed students and determined that 52% of the students disliked the music with a margin of error of 3% that was calculated at a 95% confidence level. Can the student body officers say for certain that over half of the student body dislikes the music? The confidence interval for this situation is $52\% \pm 3\%$ or 49% to 55%. The confidence level of 95% allows us to say that we are 95% confident that the true percentage is between 49% and 55%. However, it is plausible that the percentage of students who dislike the music is less than 50%. Therefore, it is probably not wise to approach the administration about changing the music just yet.

When calculating the means from several trials of a simulation, the results are normally distributed. Recall that in a normal distribution 68% of the data falls within 1 standard deviation of the mean, 95% of the data falls within two standard deviations of the mean, and 99.7% of data falls within three standard deviations of the mean. We can use the fact that 95% of the data is within 2 standard deviations of the mean to find a margin of error with a 95% confidence level.



VOCABULARY

The **margin of error** accounts for the variation in results if the study or simulation were to be conducted multiple times under the same conditions. It does account for the random selection of individuals but it does not account for bias.

A **confidence interval** provides a range of plausible values for a population parameter. It can be found by using the sample statistic and the margin of error for the confidence level desired, i.e. $\text{sample statistic} \pm \text{margin of error}$.

The **confidence level** determines how likely it is that the actual population parameter falls on the confidence interval that was calculated using the sample statistic and the margin of error.

Calculating the Margin of Error

The following formulas can be used to approximate the margin of error with a 95% confidence level:

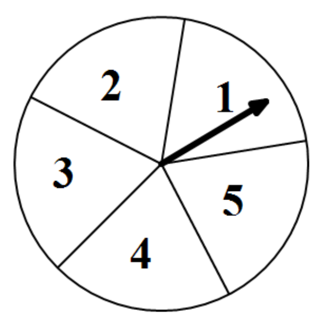
1. If you are given a sample proportion, then the margin of error can be approximated by:

   $\text{margin of error} = 2 \cdot \sqrt{\frac{\rho(1-\rho)}{n}}$, where ρ is the sample proportion and *n* is the sample size.

2. If you are given sample mean and standard deviation, then the margin of error can be approximated by: $\text{margin of error} = 2 \cdot \frac{s}{\sqrt{n}}$, where *s* is the sample standard deviation and *n* is the sample size.

Example 1:

A spinner like the one shown at the right was spun 30 times and the number it landed on was recorded as shown below.



1	2	3	4	5
$\cancel{\text{IIII}}\ \text{I}$	IIII	$\cancel{\text{IIII}}\ \text{I}$	IIII	$\cancel{\text{IIII}}\ \text{IIII}$

For each situation, find the sample proportion, the margin of error for a 95% confidence level, the 95% confidence interval for the population proportion, and determine if the theoretical proportion would be within the confidence interval found.

a. the probability of the spinner landing on 2
b. the probability of the spinner landing on 3
c. the probability of the spinner landing on 5

Answer:
The theoretical probability for each number will be the same. It can be found by using the formula:
$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$. In the case of the spinner, there is one number for the favorable outcome and 5 numbers for the total number of outcomes, therefore, the theoretical probability will be $P(E) = \frac{1}{5} = 0.2$.

a. $P(2)=\dfrac{\text{number of favorable outcomes}}{\text{total number of trials}}=\dfrac{4}{30}\approx 0.133$	The spinner landed on 2 four times out of 30 trials.
$\text{margin of error}=2\cdot\sqrt{\dfrac{0.133(1-0.133)}{30}}$ $\text{margin of error}\approx 0.124$	We have a sample statistic, so we will use the formula $\text{MOE}=2\cdot\sqrt{\dfrac{\rho(1-\rho)}{n}}$ to approximate the margin of error.
0.133 ± 0.124 $0.133-0.124=0.009$ $0.133+0.124=0.257$ The confidence interval is from 0.009 to 0.257.	The confidence interval is found by adding and subtracting the margin of error from the sample statistic.
The theoretical probability of 0.2 would fall in the confidence interval from 0.009 to 0.257.	

b. $P(3)=\dfrac{\text{number of favorable outcomes}}{\text{total number of trials}}=\dfrac{6}{30}=0.2$	The spinner landed on 3 six times out of 30 trials.
$\text{margin of error}=2\cdot\sqrt{\dfrac{0.2(1-0.2)}{30}}$ $\text{margin of error}\approx 0.146$	We have a sample statistic, so we will use the formula $\text{MOE}=2\cdot\sqrt{\dfrac{\rho(1-\rho)}{n}}$ to approximate the margin of error.
0.2 ± 0.146 $0.2-0.146=0.054$ $0.2+0.146=0.346$ The confidence interval is from 0.054 to 0.346.	The confidence interval is found by adding and subtracting the margin of error from the sample statistic.
The theoretical probability of 0.2 would fall in the confidence interval from 0.054 to 0.346.	

c. $P(5)=\dfrac{\text{number of favorable outcomes}}{\text{total number of trials}}=\dfrac{9}{30}=0.3$	The spinner landed on 5 nine times out of 30 trials.
$\text{margin of error}=2\cdot\sqrt{\dfrac{0.3(1-0.3)}{30}}$ $\text{margin of error}\approx 0.167$	We have a sample statistic, so we will use the formula $\text{MOE}=2\cdot\sqrt{\dfrac{\rho(1-\rho)}{n}}$ to approximate the margin of error.
0.3 ± 0.167 $0.3-0.167=0.133$ $0.3+0.167=0.467$ The confidence interval is from 0.133 to 0.467.	The confidence interval is found by adding and subtracting the margin of error from the sample statistic.
The theoretical probability of 0.2 would fall in the confidence interval from 0.133 to 0.467.	

Example 2:
A fast food restaurant manager wanted to determine the wait times for customers in line. He timed the customers chosen at random.

Wait Time in Minutes		
6.4	9.3	3.9
4.3	6.4	4.8
6.7	3.7	5.4
4.0	3.3	5.9
4.5	8.1	2.6
2.9	4.4	3.1
6.0	5.5	5.9
3.6	8.0	3.7
7.7	8.1	9.4
9.9	9.4	2.6

a. Find the mean and standard deviation for the sample. (Round to the nearest tenth)
b. Approximate the margin of error for a 95% confidence level and round to the nearest tenth.
c. Find the 95% confidence interval.
d. Interpret the meaning of the interval in terms of wait times for customers.

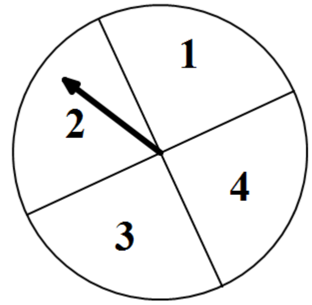
Answer:

a. The sample mean, $\bar{x}$, is 5.7. The sample standard deviation, *S*, is 2.2.	Enter the data in a graphing calculator and run the one-variable statistics.
b. margin of error $= 2 \cdot \frac{s}{\sqrt{n}}$ margin of error $= 2 \cdot \frac{2.2}{\sqrt{30}}$ margin of error ≈ 0.8	We have a sample mean and standard deviation so use the formula $\text{MOE} = 2 \cdot \frac{s}{\sqrt{n}}$ to approximate the margin of error.
c. 5.7 ± 0.8 $5.7 - 0.8 = 4.9$ $5.7 + 0.8 = 6.5$ The confidence interval is from 4.9 minutes to 6.5 minutes.	The confidence interval is found by adding and subtracting the margin of error from the sample statistic.
d. We can say with 95% confidence that the actual average wait time for a customer at the fast food restaurant is between 4.9 minutes and 6.5 minutes.	

Practice Exercises A

1. A spinner like the one shown was spun 40 times and the number it landed on was recorded as shown below.

1	2	3	4
𝍸 ||||	𝍸	𝍸 𝍸 ||	𝍸 𝍸 ||||



   For each situation, find the sample proportion, the margin of error for a 95% confidence level, the 95% confidence interval for the population proportion, and determine if the theoretical proportion would be within the confidence interval found.

   a. the probability of the spinner landing on 1
   b. the probability of the spinner landing on 2
   c. the probability of the spinner landing on 4

2. A consumer research group tested battery life of 36 randomly chosen cell phones to establish the likely battery life for the population of the same type of cell phone.

Battery Life in Hours			
55.4	63.3	72.7	70.6
50.2	85.4	85.2	83.2
72.0	69.5	65.4	65.1
55.7	73.1	47.9	72.9
55.3	58.6	81.1	58.5
64.0	83.7	73.0	74.7
80.0	73.9	75.4	58.9
61.3	69.8	83.3	61.2
63.0	63.1	85.0	57.6

   a. Find the mean and standard deviation for the sample. (Round to the nearest tenth)
   b. Approximate the margin of error for a 95% confidence level and round to the nearest tenth.
   c. Find the 95% confidence interval.
   d. Interpret the meaning of the interval in terms of battery life for this type of cell phone.

3. In a poll of 650 likely voters, 338 indicated that they planned to vote for a particular candidate.
   a. Find the sample proportion.
   b. Approximate the margin of error for a 95% confidence level.
   c. Find the 95% confidence interval.
   d. Interpret the meaning of the interval in terms of the election.

4. A recent school poll showed that 47% of respondents favored Willy B. Smart for student body president, while 51% favored Flora Bunda, with a margin of error of 4% for each poll. Can a winner be determined from the poll?

Unit 1 Cluster 3 (S.IC.5): Compare Two Treatments

Cluster 3: Make inferences and justify conclusions form sample surveys, experiments, and observational studies

1.3 Use data from a randomized experiment to compare two treatments.

1.3 Use simulations to decide if differences between parameters are significant.

Doing an experiment is the only way to prove causation. Experiments include at least one treatment group and a control group. A control group is made up of experimental units that do not receive the treatment. By comparing the results from the treatment group and the control group, we can decide if the differences are enough to convince us that the treatment was effective. We call these “statistically significant” results. Using a technique called **resampling** we can compare the results of two treatments and determine if the treatment’s effect was by chance or statistically significant. Resampling uses existing sample data from an experiment. It is assumed that the treatment has no discernible effect on the units and therefore a reassignment of units between the two groups would result in the same statistic. To perform a resampling, you combine all of the results in one data set. You conduct a simulation with several trials that uses random sampling to redistribute the data into new treatments groups. The data from each trial is compiled into a plot. The original data is then compared to the simulation data.

Example 1:
Seventy students were selected to participate in an experiment to test the effectiveness of a new homework policy. Thirty-five of the students were randomly selected to be in a group that received the new homework policy and thirty-five of the students were selected to be in a group that did not receive the new homework policy. At the end of the study, the grades of 27 students who received the new homework policy increased compared to 20 students who were in the group with no change in the homework policy. Determine if the results are significant.

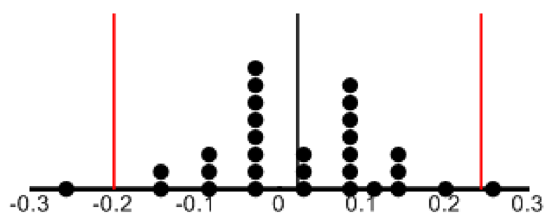
Answer:
We were not given the data, but we have the proportion of students who improved for each group. Combining this information we have 47 students out of a total of 70 students who improved. We are going to assume that this would happen regardless of whether or not the students received the new homework policy.

To simulate the situation, 47 red marbles were used to represent the students who improved and 23 blue marbles were used to represent the students who did not improve. Thirty-five marbles were drawn without replacement and the number of red marbles drawn was recorded. Thirty trials were done. The results are displayed below.

Trial	New Homework Policy		Old Homework Policy		Difference Between Proportions
	Red Marbles	Proportion	Red Marbles	Proportion	
1	25	0.714	22	0.629	0.085
2	21	0.600	26	0.743	-0.143
3	23	0.657	24	0.686	-0.029
4	28	0.800	19	0.543	0.257
5	22	0.629	25	0.714	-0.085
6	23	0.657	24	0.686	-0.029
7	25	0.714	22	0.629	0.085
8	23	0.657	24	0.686	-0.029
9	25	0.714	22	0.629	0.085

10	25	0.714	22	0.629	0.085
11	24	0.686	23	0.657	0.029
12	25	0.714	22	0.629	0.085
13	23	0.657	24	0.686	-0.029
14	24	0.686	23	0.657	0.029
15	26	0.743	21	0.600	0.143
16	23	0.657	24	0.686	-0.029
17	23	0.657	24	0.686	-0.029
18	21	0.600	26	0.743	-0.143
19	24	0.686	23	0.657	0.029
20	26	0.743	21	0.600	0.143
21	23	0.657	24	0.686	-0.029
22	25	0.714	21	0.600	0.114
23	19	0.543	28	0.800	-0.257
24	23	0.657	24	0.686	-0.029
25	22	0.629	25	0.714	-0.085
26	25	0.714	22	0.629	0.085
27	27	0.771	20	0.571	0.200
28	25	0.714	22	0.629	0.085
29	22	0.629	25	0.714	-0.085
30	26	0.743	21	0.600	0.143

The dot plot below graphs the difference between the proportion of the new policy and the old policy. The two vertical lines on the outside edge represent 2 standard deviations below and above the mean. The mean for the difference between proportions in the simulation is 0.0217.



In our original test groups the proportions were:

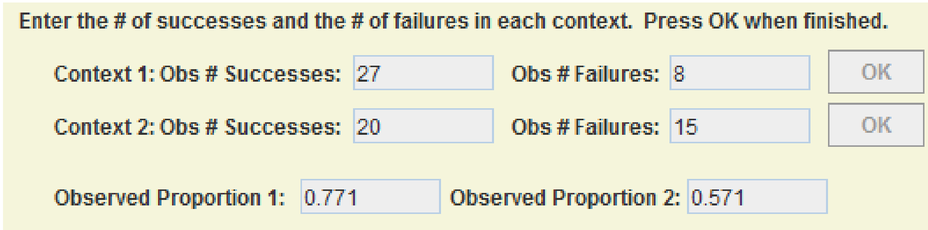
$$\text{Treatment group: } \frac{\text{number of students who improved}}{\text{total number of students}} = \frac{27}{35} \approx 0.771$$

$$\text{Control group: } \frac{\text{number of students who improved}}{\text{total number of students}} = \frac{20}{35} \approx 0.571$$

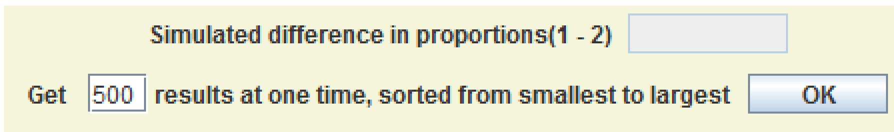
The difference was: $\text{Proportion new} - \text{Proportion old} = 0.771 - 0.571 = 0.2$.

We need to determine if a difference of 0.2 is statistically significant. Based on our simulation, it is not outside the interval that contains 95% of the data (95% of all data lies within two standard deviations of the mean in a normal distribution), therefore it is not statistically significant.

You can use a computer simulation to do more trials. The simulation can be found at: http://bcs.whfreeman.com/sris/#730892__752214__. For this simulation we are finding the difference in proportions. Use the applet that says "Difference in Proportions." Context 1 represents the students who received the new homework policy (27 successes and 8 failures) and context 2 represents the students who did not receive it (20 success and 15 failures).



Down below you can type in how many results you want the computer to do at one time. For this simulation 500 was selected.



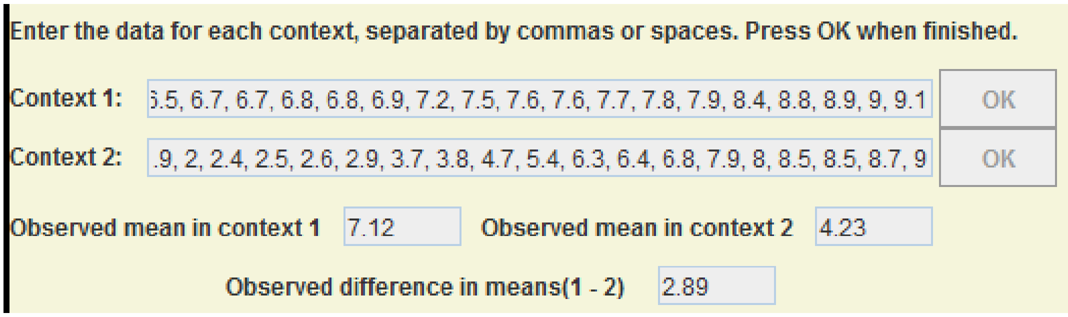
The data was exported to an excel spreadsheet. The mean was -0.00057 and the standard deviation was 0.108825. The interval containing 95% of the data is from -0.218 to 0.217. Our sample difference of 0.2 is still within this interval so it is not significant.

Example 2:
Fifty students were selected to participate in an experiment to test the effectiveness of a new shampoo that claimed to have an ingredient to make hair shinier. Twenty-five of the students were randomly selected to receive the new shampoo and twenty-five of the students were selected to receive the same shampoo but without the new ingredient. After six weeks of using the shampoo, the two groups were analyzed by hair shininess experts. The sample data is displayed in the table below.

Group	Level of shininess compared to before use of shampoo
Shampoo with ingredient	5.3, 5.5, 5.5, 5.7, 5.8, 5.9, 6.3, 6.5, 6.7, 6.7, 6.8, 6.8, 6.9, 7.2, 7.5, 7.6, 7.6, 7.7, 7.8, 7.9, 8.4, 8.8, 8.9, 9, 9.1
Shampoo without ingredient	0.3, 0.5, 0.6, 0.6, 0.6, 1.2, 1.9, 2, 2.4, 2.5, 2.6, 2.9, 3.7, 3.8, 4.7, 5.4, 6.3, 6.4, 6.8, 7.9, 8, 8.5, 8.5, 8.7, 9

The mean change in shininess of the group who used the new ingredient shampoo is 7.116 and the mean change in shininess of the group who used the regular shampoo is 4.232. Using a computer simulation with 500 trials, determine if this a significant difference.

Answer:
Go to the website http://bcs.whfreeman.com/sris/#730892__752214__. Then use the statistics applet that says "Difference in Means." Enter the data for the new ingredient shampoo group in context 1 and the data for the regular shampoo group in context 2. Then at the bottom have the simulation run 500 trials.



The mean of the differences in the simulation is -0.0287. The standard deviation of the simulation differences is 1.595. The interval that contains 95% of the data is from -1.624 to 1.566. The sample difference we began with had a difference in means of 2.89. This is beyond the interval so it is significant.

Practice Exercises A

1. Seventy-two students were selected to participate in a study to determine if computerized testing distracts students enough to lower test scores. All students were given a baseline test using paper and pencil. Thirty-six of the students were randomly selected to be in a group that used an electronic device to take a post-test. The other thirty-six students took their test using paper and pencil. After completing the post-test, the scores of twenty students using the computer were lower than the baseline test, while the scores of ten students using paper and pencil were also lower. Design a simulation to determine if the results are significant.

2. A farmer wants to know if a new enriched top soil will produce better corn plants. The farmer has 20 fields. He randomly assigns 10 fields to receive the enriched top soil and 10 fields to receive the same top soil he has used in the past. At harvest time, fields that received the enriched top soil had corn plants that had grown to mean heights of 6.5 feet, 7.9 feet, 8 feet, 7.2 feet, 7.6 feet, 6.6 feet, 8 feet, 6.3 feet, 7.8 feet, and 6.8 feet while the plants with no enriched soil grew to mean heights of 6.8 feet, 6.8 feet, 6.9 feet, 6.4 feet, 6.8 feet, 6.8 feet, 6.5 feet, 6.4 feet, 6.4 feet, and 7 feet. Using a computer simulation with 500 trials, determine if this a significant difference.

Selected Answers

Unit 3 Cluster 1 (G.SRT.9)

Practice Exercises A (Pg. 6)

1. 222.332 ft^2
3. 5.290 cm^2
5. 17.973 ft^2
7. 128.079 m^2
9. 204.985 ft

Unit 3 Cluster 1 (G.SRT.10 and G.SRT.11)

Practice Exercises A (Pg. 10–11)

1. $C = 110°$, $a \approx 12.856$, $c \approx 18.794$
3. $C = 75°$, $a \approx 4.532$, $c \approx 5.054$
5. $A = 77°$, $b \approx 12.686$, $c \approx 9.426$
7. $B \approx 28.822°$, $C \approx 111.178°$, $c \approx 29.013$
9. $A \approx 99.290°$, $C \approx 30.710°$, $a \approx 38.649$
11. $B \approx 41.328°$, $C \approx 89.672°$, $c \approx 42.400$
13. $B \approx 45.805°$, $C \approx 54.196°$, $b \approx 12.376$
15. $A \approx 38.647°$, $B \approx 105.353°$, $b \approx 26.249$ or $A \approx 141.353°$, $B \approx 2.647°$, $b \approx 1.257$
17. no triangle can be formed
19. $A \approx 94.867°, B \approx 47.133°$, $a \approx 33.987$ or $A \approx 9.133°, B \approx 132.867°$, $a \approx 5.414$
21. $B \approx 64.534°, C \approx 52.466°$, $b \approx 10.133$

Practice Exercises B (Pg. 14)

1. $A \approx 26.540°, B \approx 126.460°$, $c \approx 5.080$
3. $A \approx 71.992°, C \approx 68.008°$, $b \approx 54.072$
5. $A \approx 45.530°, B \approx 92.470°$, $c \approx 4.688$
7. $A \approx 22.332°, B \approx 108.21°$, $C \approx 49.458°$
9. $A \approx 27.660°, B \approx 40.536°$, $C \approx 11.804°$
11. $A \approx 34.960°, B \approx 46.826°$, $C \approx 98.213°$
13. Law of Sines, $B \approx 77.109°$ or $B \approx 102.891°$
15. Law of Cosines, $a \approx 5.962$
17. Law of Sines, $c \approx 1.051$
19. Law of Cosines, $C \approx 39.974°$

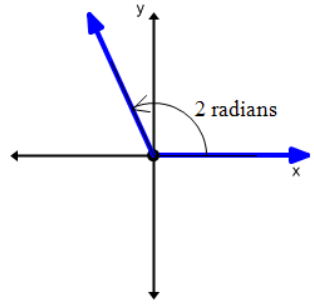
Practice Exercises C (Pg. 17–18)

1. 13 miles or 2 miles
3. 4.8 miles
5. distance from A 10.1 miles, distance from B 13.6 miles
7. 93.2°
9. 28.9 feet
11. $87°$, it is leaning
13. 100.2°
15. 214.4 yards

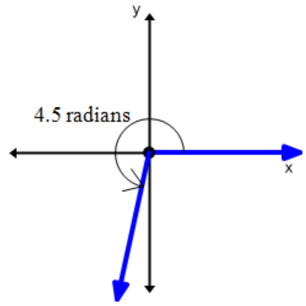
Unit 3 Cluster 2 (F.TF.1, F.TF.2, and F.TF.3)

Practice Exercises A (Pg. 21)

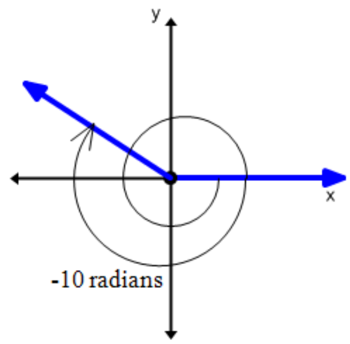
1.



3.



5.



Practice Exercises B (Pg. 29)

1. $0, 2\pi$
3. $\frac{\pi}{4}$
5. $\frac{\pi}{2}$
7. $\frac{3\pi}{4}$
9. π
11. $\frac{5\pi}{4}$
13. $\frac{3\pi}{2}$
15. $\frac{7\pi}{4}$

Practice Exercises C (Pg. 31)

1. $-\frac{\sqrt{2}}{2}$	3. $-\frac{\sqrt{3}}{2}$
5. $\sqrt{3}$	7. -1
9. $-\frac{1}{2}$	11. 1
13. $-\frac{1}{2}$	15. $-\frac{\sqrt{3}}{3}$
17. -1	19. $-\frac{\sqrt{2}}{2}$
21. undefined	23. $-\frac{1}{2}$
25. j	27. c
29. a	31. f
33. g	35. p
37. neither	39. negative
41. positive	43. positive
45. negative	47. negative

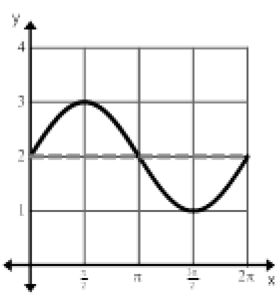
Practice Exercises D (Honors) (Pg. 33)

1. $-\sqrt{3}$	3. $-\frac{2}{\sqrt{3}}$
5. -2	7. $\sqrt{3}$
9. $-\frac{2}{\sqrt{3}}$	11. $\frac{2}{\sqrt{3}}$

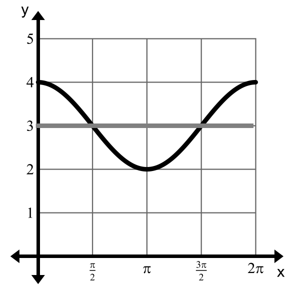
Unit 3 Clusters 2 & 3 (F.TF.2 and F.TF.5)

Practice Exercises A (Pg. 38–39)

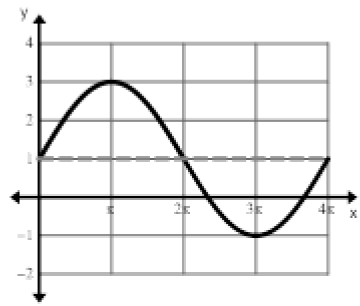
1. Amplitude: 1
   Period: 2π



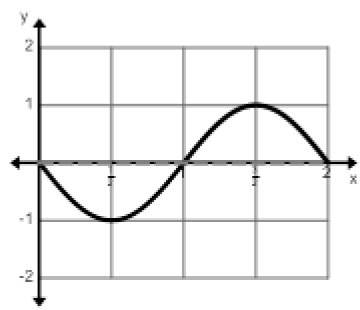
3. Amplitude: 1
   Period: 2π



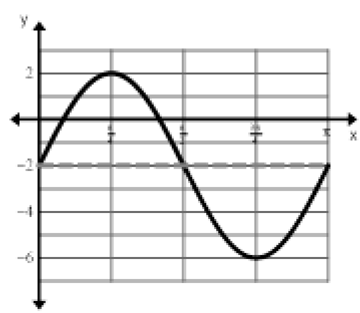
5. Amplitude: 2
   Period: 4π



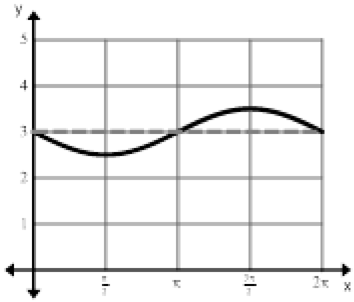
7. Amplitude: 1
   Period: 2



9. Amplitude: 4
   Period: π



11. Amplitude: $\frac{1}{2}$
Period: 2π



13. Amplitude: 2
Period: π
Midline: $y=-1$
$f(x)=-2\sin(2x)-1$

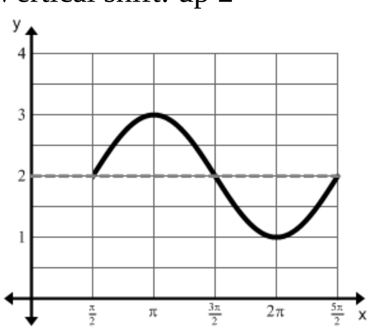
15. Amplitude: 1
Period: 6π
Midline: $y=3$
$f(x)=\sin\left(\frac{1}{3}x\right)+3$

17. Amplitude: 4
Period: 8
Midline: $y=1$
$f(x)=4\cos\left(\frac{\pi}{4}x\right)+1$

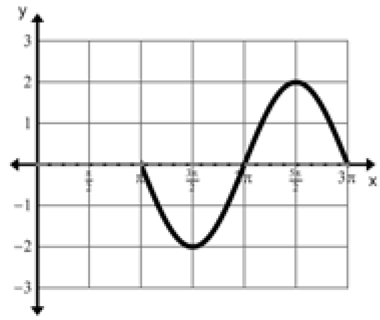
19. Amplitude: 2
Period: 2
Midline: $y=-3$
$f(x)=-2\cos(\pi x)-3$

Practice Exercises B (Pg. 41)

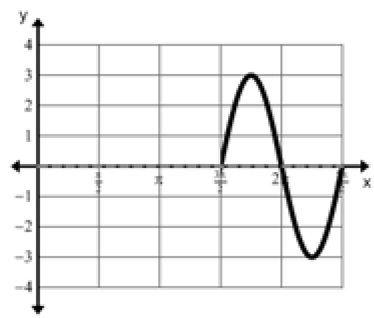
1. Amplitude: 1
Period: 2π
Phase shift: right $\dfrac{\pi}{2}$
Vertical shift: up 2



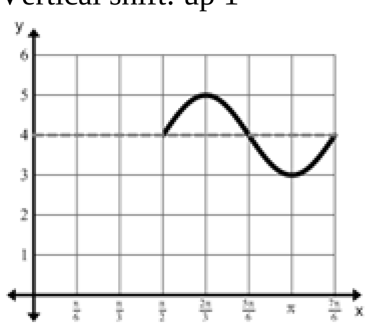
3. Amplitude: 2
Period: 2π
Phase shift: right π
Vertical shift: none



5. Amplitude: 3
Period: π
Phase shift: right $\dfrac{3\pi}{2}$
Vertical shift: none



7. Amplitude: 1
Period: $\dfrac{2\pi}{3}$
Phase shift: right $\dfrac{3\pi}{2}$
Vertical shift: up 1

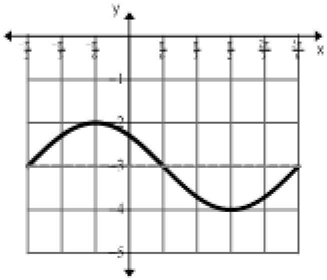


9. Amplitude: 1

Period: $\frac{4\pi}{3}$

Phase shift: left $\frac{\pi}{2}$

Vertical shift: down 3



Practice Exercises C (Pg. 43)

1. $f(t) = 3.6\cos\left(\frac{\pi}{4}t\right)$

3. $f(t) = -32\cos\left(\frac{\pi}{3}t\right)$

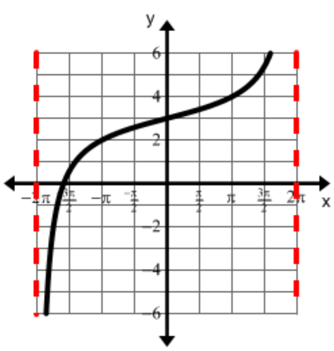
5. $f(x) = \sin(40{,}000\pi x)$

Practice Exercises D (Honors) (Pg. 46)

1. Period: 2π

Asymptotes: $x = \pm 2\pi$

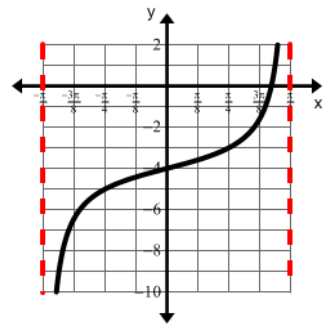
y-intercept: $(0,3)$



3. Period: π

Asymptotes: $x = \pm\frac{\pi}{2}$

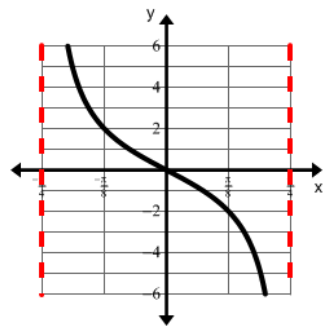
y-intercept: $(0,-4)$



5. Period: $\frac{\pi}{2}$

Asymptotes: $x = \pm\frac{\pi}{4}$

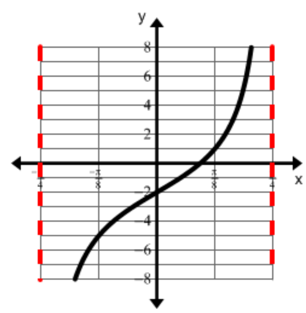
y-intercept: $(0,0)$



7. Period: $\frac{\pi}{2}$

Asymptotes: $x = \pm\frac{\pi}{4}$

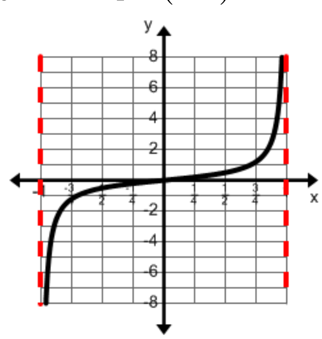
y-intercept: $(0,-2)$



9. Period: 2

Asymptotes: $x = \pm 1$

y-intercept: $(0,0)$



Unit 3 Clusters 2 and 3 Honors (F.TF.4)

Practice Exercises A (Pg. 49)

1. $\sin\left(-\frac{2\pi}{3}\right)=-\frac{\sqrt{3}}{2}$, $\cos\left(-\frac{2\pi}{3}\right)=-\frac{1}{2}$
3. $\sin\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}$, $\cos\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}$
5. $\sin\left(-\frac{5\pi}{6}\right)=-\frac{1}{2}$, $\cos\left(-\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$
7. $\sin\left(\frac{13\pi}{6}\right)=\frac{1}{2}$
9. $\sin\left(\frac{13\pi}{2}\right)=1$
11. $\sin\left(\frac{61\pi}{3}\right)=\frac{\sqrt{3}}{2}$
13. $\sin\left(\frac{24\pi}{4}\right)=\frac{\sqrt{2}}{2}$
15. $\sin\left(\frac{7\pi}{2}\right)=1$

Unit 3 Cluster 3 Honors (F.TF.6 and F.TF.7)

Practice Exercises A (Pg. 54)

1. $\frac{\pi}{6}$
3. $\frac{\pi}{2}$
5. $-\frac{\pi}{4}$
7. $-\frac{\pi}{2}$
9. $\frac{\pi}{6}$
11. π
13. $\frac{2\pi}{3}$
15. $\frac{\pi}{3}$
17. 0
19. $-\frac{\pi}{6}$

Practice Exercises B (Pg. 58)

1. 0.8
3. 7
5. $\frac{\sqrt{2}}{2}$
7. $\frac{\pi}{6}$
9. $-\frac{\pi}{2}$
11. $-\frac{\pi}{4}$
13. $-\frac{\pi}{3}$
15. $\frac{\pi}{6}$
17. $\frac{\sqrt{2}}{2}$
19. $\frac{1}{2}$
21. $\frac{24}{25}$
23. $-\frac{4}{3}$
25. $\frac{15}{17}$
27. $\frac{3}{\sqrt{7}}$
29. $\frac{x}{\sqrt{1+x^2}}$
31. $\frac{1}{\sqrt{x^2-1}}$
33. $\frac{x}{\sqrt{x^2+4}}$
35. $\frac{3}{x}$

Unit 3 Cluster 3 Honors Solving Trigonometric Equations

Practice Exercises A (Pg. 61)

1. $x=0$ or π
3. $x=\frac{3\pi}{4}$ or $\frac{7\pi}{4}$
5. $x=\frac{5\pi}{6}$ or $\frac{7\pi}{6}$
7. $x=\frac{4\pi}{3}$ or $\frac{5\pi}{3}$
9. $x=\frac{\pi}{6}$ or $\frac{7\pi}{6}$
11. $x=\pi$
13. $x=\frac{5\pi}{6}$ or $\frac{7\pi}{6}$
15. $x=\frac{3\pi}{4}$ or $\frac{7\pi}{4}$
17. $x=\frac{3\pi}{2}$
19. 0.366 or 2.805
21. 1.016 or 4.158
23. 0.406 or 3.548
25. 1.966 or 5.107
27. 1.400 or 4.883

Practice Exercises B (Pg. 64)

1. $x=\dfrac{4\pi}{3}$ or $\dfrac{5\pi}{3}$

3. $x=\dfrac{3\pi}{4}$ or $\dfrac{5\pi}{4}$

5. $x=\dfrac{2\pi}{3}$ or $\dfrac{4\pi}{3}$

7. $x=\dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$

9. $x=0,\ \dfrac{\pi}{6},\ \dfrac{5\pi}{6},\ \pi,\ \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$

11. $x=0,\ \dfrac{\pi}{4},\ \pi$ or $\dfrac{5\pi}{4}$

13. $x=\dfrac{\pi}{4},\ \dfrac{3\pi}{4},\ \dfrac{5\pi}{4}$ or $\dfrac{7\pi}{4}$

15. $x=\dfrac{\pi}{3},\ \dfrac{2\pi}{3},\ \dfrac{4\pi}{3}$ or $\dfrac{5\pi}{3}$

17. $x=0,\ \dfrac{\pi}{4},\ \pi,$ or $\dfrac{7\pi}{4}$

19. $x=0,\ \dfrac{\pi}{2},\ \pi$ or $\dfrac{3\pi}{2}$

21. $x=\dfrac{7\pi}{6},\ \dfrac{3\pi}{2}$ or $\dfrac{11\pi}{6}$

23. $x=0$ or π

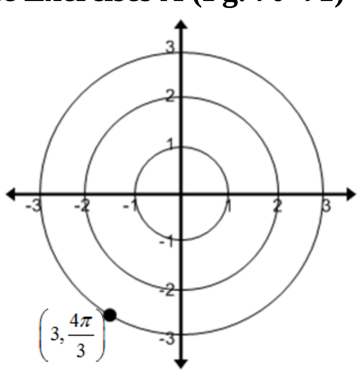
25. 0.0095 seconds

27. 4.111 hours and 8.222 hours

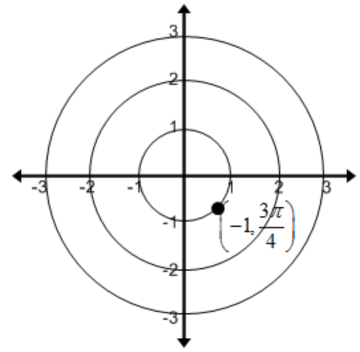
Honors Defining and Using Polar Coordinates

Practice Exercises A (Pg. 70–71)

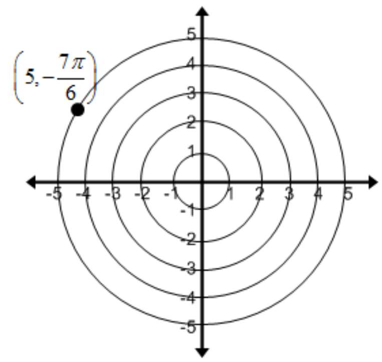
1.



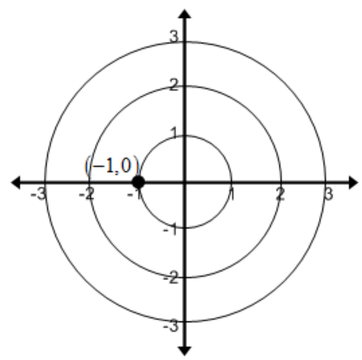
3.



5.



7.



9.
   a. $\left(5,\dfrac{13\pi}{6}\right)$
   b. $\left(-5,\dfrac{7\pi}{6}\right)$
   c. $\left(5,-\dfrac{11\pi}{6}\right)$

11.
   a. $\left(10,\dfrac{11\pi}{4}\right)$
   b. $\left(-10,\dfrac{7\pi}{4}\right)$
   c. $\left(10,-\dfrac{5\pi}{4}\right)$

13. a. $\left(12, \frac{13\pi}{4}\right)$

b. $\left(-12, \frac{\pi}{4}\right)$

c. $\left(12, -\frac{3\pi}{4}\right)$

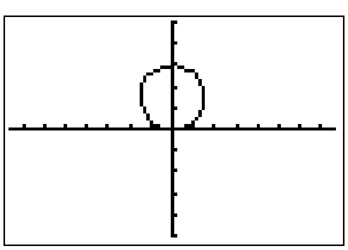
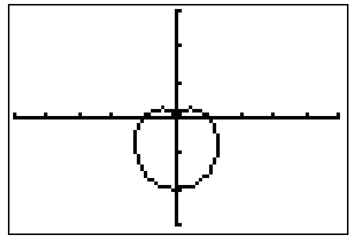
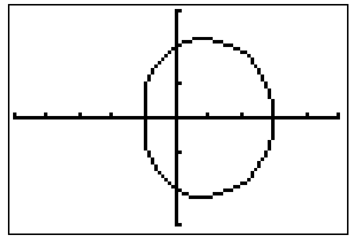
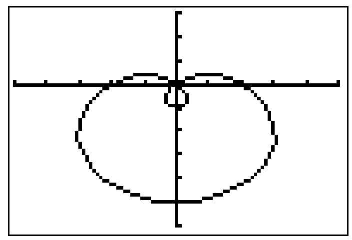
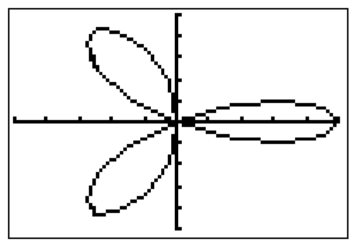
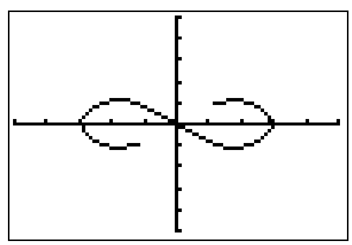
15. a. $\left(3, \frac{5\pi}{2}\right)$

b. $\left(-3, \frac{3\pi}{2}\right)$

c. $\left(3, -\frac{3\pi}{2}\right)$

17. D
19. A
21. B
23. C
25. $(0,4)$
27. $\left(1, \sqrt{3}\right)$
29. $\left(3\sqrt{2}, -3\sqrt{2}\right)$
31. $\left(-\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$
33. $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$
35. $\left(4, \frac{5\pi}{3}\right)$
37. $\left(2, \frac{7\pi}{6}\right)$
39. $(3, \pi)$

Practice Exercises B (Pg. 75)

1. $r = \frac{-3}{\cos\theta}$
3. $r = \frac{4}{\sin\theta}$
5. $r = \frac{5}{2\cos\theta - 3\sin\theta}$
7. $r = \frac{8}{\cos\theta + 5\sin\theta}$
9. $r^2 = 25$
11. $r = 6\cos\theta$
13. $r = \frac{3\sin\theta}{\cos^2\theta}$
15. $r = \frac{4\sin\theta - 8\cos\theta}{\cos^2\theta}$
17. $r = -6\cos\theta - 6\sin\theta$
19. 
21. 
23. 
25. 
27. 
29. 

Honors Unit Complex Numbers in Polar Form

Practice Exercises A (Pg. 80–81)

1. $3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
3. $2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$
5. $5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

7. $4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

9. $6\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

11. $\sqrt{13}\left(\cos 4.124+i\sin 4.124\right)$

13. $2\sqrt{3}+2i$

15. $-4-4\sqrt{3}i$

17. $2\sqrt{2}-2\sqrt{2}i$

19. $7i$

21. -3

23. $\frac{\sqrt{3}}{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$; $-0.837+0.224i$

25. $2\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$; $0.518-1.932i$

27. $3\left(\cos\frac{41\pi}{30}+i\sin\frac{41\pi}{30}\right)$; $-1.220-2.741i$

29. $\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)$; $0.696+0.123i$

31. $k=0;\ \sqrt[3]{2}\left(\cos\frac{11\pi}{18}+i\sin\frac{11\pi}{18}\right)$

$k=1;\ \sqrt[3]{2}\left(\cos\frac{23\pi}{18}+i\sin\frac{23\pi}{18}\right)$

$k=2;\ \sqrt[3]{2}\left(\cos\frac{35\pi}{18}+i\sin\frac{35\pi}{18}\right)$

33. $k=0;\ 2\sqrt[3]{4}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

$k=1;\ 2\sqrt[3]{4}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$

$k=2;\ 2\sqrt[3]{4}\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)$

35. $k=0;\ \sqrt[4]{108}\left(\cos\frac{\pi}{24}+i\sin\frac{\pi}{24}\right)$

$k=1;\ \sqrt[4]{108}\left(\cos\frac{13\pi}{24}+i\sin\frac{13\pi}{24}\right)$

$k=2;\ \sqrt[4]{108}\left(\cos\frac{25\pi}{24}+i\sin\frac{25\pi}{24}\right)$

$k=3;\ \sqrt[4]{108}\left(\cos\frac{37\pi}{24}+i\sin\frac{37\pi}{24}\right)$

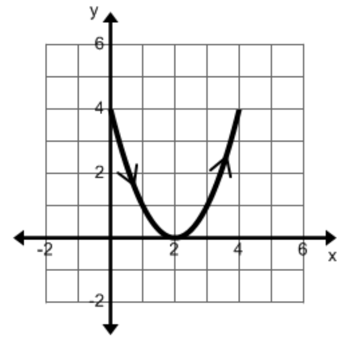
37. $k=0;\ 2\left(\cos 0+i\sin 0\right);\ 2$

$k=1;\ 2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right);\ -1+\sqrt{3}i$

$k=2;\ 2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right);\ -1-\sqrt{3}i$

Honors Unit Parametric Equations

Practice Exercises A (Pg. 86–87)

1.


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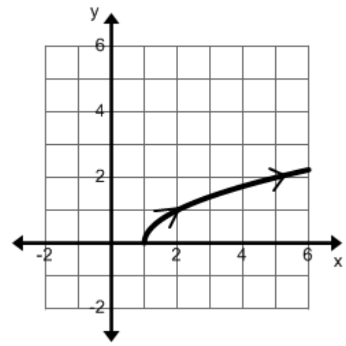

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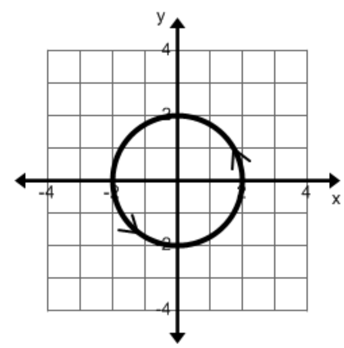

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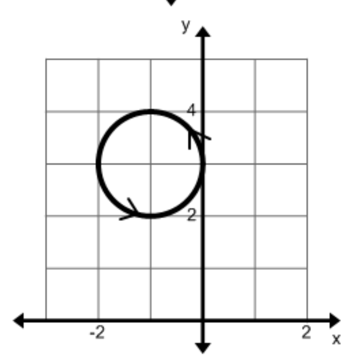
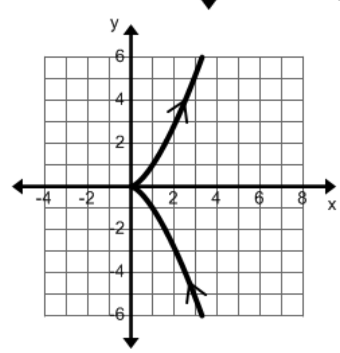


9.


11. $y = x - 1$

13. $y = x^2 - 2x + 3$

15. $y = (x+4)^2$

17. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

19. $\dfrac{(x-1)^2}{9} + \dfrac{(y+1)^2}{4} = 1;\ 1 \le x \le 4$

21. $x = 270\cos(48)t$

$y = 270\sin(48)t - 16t^2 + 4$

The arrow hits the ground after about 12.560 seconds.
The arrow lands about 2269.241 feet from where it starts.

23. $x = 65\cos(36)t$

$y = 205\sin(35)t - 16t^2 + 0.5$

The ball will make it over the goal post.

25. $x = 150\cos(35)t$

$y = 150\sin(35)t - 16t^2 + 3$

The ball is in flight for approximately 5.412 seconds.
The ball travel about 664.978 feet horizontally.

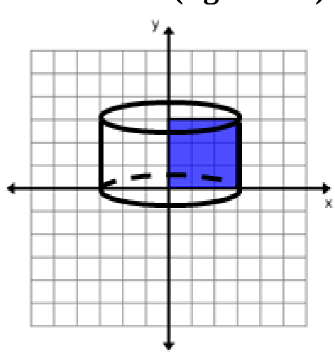
Unit 4 Cluster 7 (G.GMD.4) Two and Three Dimensional Objects

Practice Exercises A (Pg. 92)

1. circle
3. ellipse
5. triangle
7. pentagon
9. rectangle
11. pentagon
13. circle
15. hexagon
17. heptagon

Practice Exercises B (Pg. 94–95)

1.



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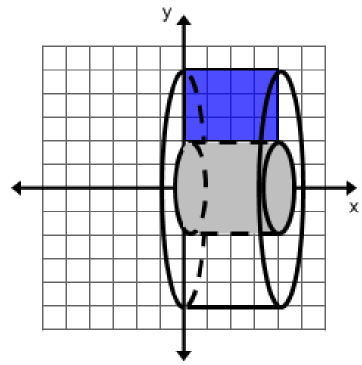
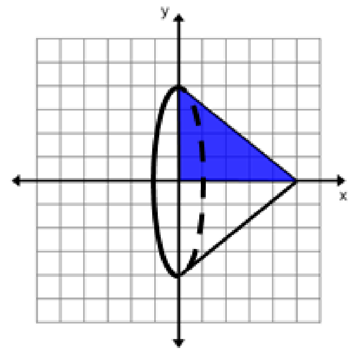


5.



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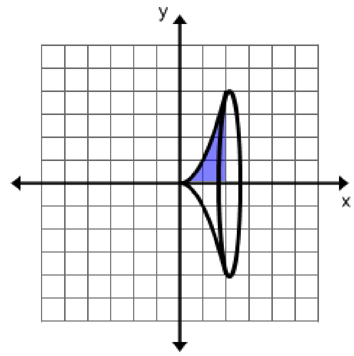
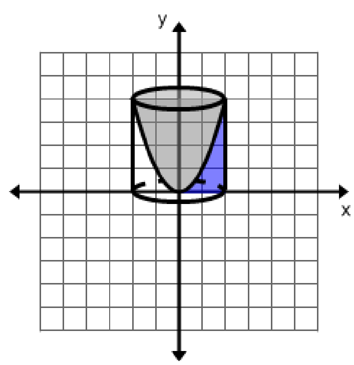


9.



Unit 4 Cluster 8 (G.MG.1, 2, and 3) Modeling with Geometry

Practice Exercises A (Pg. 98)

1. 31.269 in^2
3. 1222 bricks

Practice Exercises B (Pg. 100)

1. Texas has a higher population density with 97.025 people per square foot.
3. cedar plank 0.0133 density
   Oregon pine 0.0192 density
   Oregon pine is more dense

Unit 1 Cluster 2 (S.IC.1) Inferences

Practice Exercises A (Pg. 104–105)

1. a. all eligible voters in Utah
   b. all eligible voters in 15 state house districts
   c. percentage of people who vote in Utah
3. a. all Utahns over the age of 12
   b. 1200 Utahns over the age of 12
   c. the average amount of time they spend exercising
5. it is a convenience sampling and it is biased because they are conducting the survey outside of an arts program so arts supporters will be overrepresented

Unit 1 Cluster 2 (S.IC.2) Simulation

Practice Exercises A (Pg. 109)

1. Answers will vary. In 50 trials, 14 had exactly 34 students or 28% of the time.
3. Answers will vary. In 50 trials, 1 had a 8 female students, which is a probability of 2%.

Unit 1 Cluster 3 (S.IC.3 and S.IC.6) Surveys, Experiments, Observations, and Evaluation of Reports

Practice Exercises A (Pg. 111)

1. survey, if the sample size is large enough the results can be applied to the population
3. experiment, if the sample size is large enough the results can be applied to the population
5. survey, if the new customers were selected randomly then the results can be applied to all new customers

Practice Exercises B (Pg. 113)

1. Answers will vary

Unit 1 Cluster 1 (S.ID.4) Normal Distribution

Practice Exercises A (Pg. 116)

1. a. 200 to 800
   b. approximately 6.7%
   c. approximately 83.5%
3. a. 71.22 inches to 86.78 inches
   b. approximately 30.6%
   c. approximately 24.2%

Unit 1 Cluster 3 (S.IC.4) Margin of Error

Practice Exercises A (Pg. 121)

1. a. sample proportion: 0.225
   margin of error: 0.132
   confidence interval: 0.093 to 0.357
   the theoretical probability of 0.25 would be in the confidence interval
   b. sample proportion: 0.125
   margin of error: 0.105
   confidence interval: 0.02 to 0.23
   the theoretical probability of 0.25 would not be in the confidence interval
   c. sample proportion: 0.35
   margin of error: 0.151
   confidence interval: 0.199 to 0.501
   the theoretical probability of 0.25 would be in the confidence interval
3. a. 52%
   b. 3.9%
   c. 48.1% to 55.9%
   d. It is plausible that the candidate has less than 50% of the vote. It is also plausible that the candidate has more than 50% of the vote so the election is too close to call.

Unit 1 Cluster 3 (S.IC.5) Compare Two Treatments

Practice Exercises A (Pg. 125)

1. Answers will vary.
   Example: the difference between the two proportions is 0.278. After running a simulation the sample proportion difference is -0.008 and the standard deviation is 0.117. The interval containing 95% of the data is -0.242 to 0.226. The sample difference is outside of this interval so it is statistically significant.